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Analytic functions with finite Dirichlet integrals

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1. In the classification theory of Riemann surfaces (cf. e.g. Sario-Nakai [2]), the problem whether the inclusion $O_{AD}\subset O_{ABD}$ is strict or not had long been open and only recently the identity $O_{AD}=O_{ABD}$ is established by an elaborate work [1] of Sakai. On the other hand, Uy [4] also recently proved the following interesting theorem: If E is an arbitrary compact subset of the complex plane C with positive area, then there exists a nonconstant bounded analytic function $\phi(z)$ on $C-E$ satisfying the Lipschitz condition on $C-E$. We first *remark* here that the above theorem implies the identity $O_{AD}=O_{ABD}$. In fact, suppose there exists a nonconstant analytic function f on a Riemann surface R with the finite Dirichlet integral $D_R(f)=\iint_R |f'(z)|^2 dxdy<+\infty$, i.e. $f\in AD(R)-C$. The image region $f(R)$ has a finite area since $D_R(f)<+\infty$, and a fortiori $C-f(R)$ has a positive area (and in fact an infinite area). Therefore we can find a compact subset E with positive area in $C-f(R)$. Let $\phi(z)$ be the function in the above theorem associated with E. It is readily checked that $\phi\circ f\in ABD(R)-C$, and we have seen the inclusion $O_{AD}\supset O_{ABD}$. This with the trivial inclusion $O_{AD}\subset O_{ABD}$ implies the identity $O_{AD}=O_{ABD}$.

One step further Sakai [1] proved that $ABD(R)$ is dense in $AD(R)$ with respect to the Dirichlet seminorm $D(\cdot)^{1/2}$. By observing the proof of $O_{AD}=O_{ABD}$ mentioned above, we naturally come across the question (suggested to the author by Professor Nakai) whether there exists a sequence $\{\phi_n\}$ on C such that $\phi_n\circ f \in ABD(R)$ and $\{\phi_n\}$ converges to the identity function on $f(R)$ so that the sequence $\{\phi_n\circ f\}$ gives the desired approximation of the given $f\in AD(R)$. The *purpose* of this note is to prove the following theorem by which the above procedure is certainly possible.

THEOREM. *Suppose that a closed set E in the complex plane C satisfies the condition*

$$\limsup_{r\to\infty} \frac{m(E\cap\{r<|z|<2r\})}{r^2}>0 \tag{1}$$

with m the Lebesgue measure on C. Then there exists a sequence of functions $\{\phi_n(z)\}$ satisfying the following three conditions: