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Some sums involving Farey fractions I

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1. Introduction.

It is our aim in this paper to give some refinements of theorems proved by Hall [6] and the first author [9] on some sums involving Farey fractions.

Let F_n ($n \in \boldsymbol{N}$) be the Farey series of order n, that is, the set of all irreducible fractions between 0 and 1 with denominators $\leqq n$ and arranged in the increasing order of magnitude: $F_n = \{h/k \mid 0 \leqq h \leqq k \leqq n,\ (h, k)=1\}$; for any term h/k (<1) of F_n we denote by h'/k' its successor in F_n and by Q_n the set of all pairs (k, k') of the denominators of such consecutive fractions in F_n. For any function $f: \boldsymbol{N} \times \boldsymbol{N} \to \boldsymbol{C}$, writing

$$S_n = \sum_{(k,k') \in Q_n} f(k, k'),$$

Lehner and Newman [11] proved the sum formula (see also Mitsui [14], pp. 106-109)

$$S_n = f(1, 1) + \sum_{r=2}^{n} \sum_{\substack{k=1 \\ (k,r)=1}}^{r} \{f(k, r) + f(r, k) - f(k, r-k)\}. \tag{1}$$

The interest in this formula arises due to the fact that a sum involving Farey fractions is transformed into one which does not. Lehner-Newman [11] and the first author [9] discussed, among other things, the applications of the sum formula (1) to the evaluation of certain infinite series. Recently, the second and third named authors [18] made use of an extension of the sum formula (1) (to be found in Apostol [1], p. 111) to proving several identities involving Riemann's zeta-function and, in particular, those of Briggs, Chowla, Kempner and Mientka [2], Gupta [5], Hans and Dumir [7] and Williams [21]. In section 2 of this paper we state refinements of the first author's sharpenings of Hall's results [6] and establish some preliminary results for the proofs of these results. The preliminary results obtained in section 2 also enable us to sharpen various

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