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On the Brun-Titchmarsh theorem

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1. Introduction.

Let a and q be relatively prime positive integers and let $\pi(x, q, a)$ stand for the number of primes $p\leqq x$ congruent to a mod q. The prime number theorem of Siegel and Walfisz (see Prachar's book [**31**]) states that

$$\pi(x;q, a)=\frac{Lix}{\varphi(q)}+O(x\exp(-A(\log x)^{1/2}))$$

uniformly for $q\leqq(\log x)^{B}$ where B is any positive constant and $A=A(B)>0$. This theorem has the defect that it holds only for relatively small values of q. The Extended Riemann Hypothesis yields (see Titchmarsh [**34**])

$$\pi(x;q, a)=\frac{Lix}{\varphi(q)}+O(x^{1/2}\log x)$$

uniformly for $q\leqq x^{1/2}(\log x)^{-2}$, but even this is not always sufficient. H.L. Montgomery conjectures that

$$\pi(x;q, a)=\frac{Lix}{\varphi(q)}+O\Big(\Big(\frac{x}{q}\Big)^{1/2+\varepsilon}\Big)$$

uniformly for all $q<x^{1-\varepsilon}$ (actually Montgomery formulated the conjecture for $\Psi(x;q, a)$, see [**24**], the above version follows easily by partial summation). Here and in what follows ε stands for any positive constant to be regarded as being small and not necessarily the same at each occurrence; the constants implied in the symbols 0 and $\ll$ depend at most on ε.

In 1930 E.C. Titchmarsh [**34**] used Brun's sieve to prove that if $q<x^{1-\varepsilon}$ then

$$\pi(x;q, a)\ll\frac{x}{\varphi(q)\log x}.$$

This bound represents the true order of magnitude of $\pi(x;q, a)$ in the whole range $q<x^{1-\varepsilon}$. Although Titchmarsh' result is much less precise than the hypothetical asymptotic formula of Montgomery it has been recognized to be equally fruitful in various problems. Titchmarsh himself applied his result for