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A remark on non-enlargable Lie algebras

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Let N be a connected, non-compact, separable, C^{∞} manifold of finite dimension, and let $\boldsymbol{\Gamma}(T_N)$ be the Lie algebra of all C^{∞} vector fields on N. In this short note, we shall remark the following:

THEOREM. *There is no "infinite dimensional Lie group" with the Lie algebra* $\boldsymbol{\Gamma}(T_N)$.

The above result shows that Lie's third theorem does not hold in a sense in case of infinite dimensional Lie algebras, but of course it depends how we define the concept of infinite dimensional Lie groups. (See also § 3 below.) Thus to give a precise statement of the above theorem we have to fix at first the meaning of "infinite dimensional Lie groups". However, since the result that we want to obtain is a negative one, we shall fix here the definition as wide as possible.

§ 1. Definition of infinite dimensional Lie groups.

Let G be an abstract group. As usual, $G^{\boldsymbol{R}}$ denotes the group of all mappings of $\boldsymbol{R}$ into G, where the group operations are defined pointwisely. By $G_e^{\boldsymbol{R}}$ we denote the subgroup consisting of all $X \in G^{\boldsymbol{R}}$ such that $X(0)=e$, the identity. For each $g \in G$, $X \in G_e^{\boldsymbol{R}}$ we denote by $A(g)X$ an element of $G_e^{\boldsymbol{R}}$ defined by $(A(g)X)(t)=gX(t)g^{-1}$. A is an action of G on $G_e^{\boldsymbol{R}}$, which will be called the *adjoint action*.

A structure of an infinite dimensional Lie group on G is a triple $\{\mathcal{S}, \mathfrak{g}, \pi\}$ of an adjoint invariant subgroup $\mathcal{S}$ of $G_e^{\boldsymbol{R}}$ such that if $g(t) \in \mathcal{S}$ then $g(t+s)g(s)^{-1} \in \mathcal{S}$ for any s, an infinite dimensional topological Lie algebra $\mathfrak{g}$ and a homomorphism π of $\mathcal{S}$ *onto* the underlying additive group of $\mathfrak{g}$ satisfying the following:

(a) For every $g \in G$, there is an automorphism $\mathrm{Ad}(g)$ of $\mathfrak{g}$ such that $\pi(A(g)X)=\mathrm{Ad}(g)\pi(X)$.

(b) For every $X \in \mathcal{S}$ and $v \in \mathfrak{g}$, the mapping $t \mapsto \mathrm{Ad}(X(t))v$ is of class C^{∞} such that $d/dt|_{t=0}\ \mathrm{Ad}(X(t))v=[u, v]$, where $u=\pi(X)$ and $[\ ,\ ]$ is the Lie bracket product defined on $\mathfrak{g}$. (See [2], [3] for the definition of differentiability.)

(c) There is a mapping $\exp: \mathfrak{g} \to G$ such that for every $u \in \mathfrak{g}$, $X(t)=\exp tu$ is an element of $\mathcal{S}$, $\{\exp tu\,;\, t \in \boldsymbol{R}\}$ is a one parameter subgroup of G and $\pi(X)$