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The critical values of certain zeta functions associated with modular forms of half-integral weight

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Introduction.

The main theme of this paper is the algebraicity of two types of numbers occurring in connection with modular forms. One is the values at certain integers or half-integers of a Dirichlet series

$$D(s, f, g)=\sum_{n=1}^{\infty} a(n)b(n)n^{-s}$$

obtained from two modular forms

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2\pi inz} \quad \text{and} \quad g(z)=\sum_{n=0}^{\infty} b(n)e^{2\pi inz},$$

and the other is the inner product $\langle f, g\rangle$ of f and g, when they have the same weight. These have been treated in our previous papers [**12**], [**13**], and [**14**] for the forms f and g of integral weight. Therefore the present investigation concerns the cases in which either or both of f and g have half-integral weight.

To describe the nature of the problems as well as of the results, we let m' and m denote the weights of f and g, respectively, which are positive elements of $2^{-1}\boldsymbol{Z}$. If $m=m'$, there is a well known relation between $\langle f, g\rangle$ and the residue of $D(s, f, g)$ at $s=m$, and therefore the second object has a character similar to the first one. Setting this residue aside, we restrict, for some natural reasons, the study of the values of $D(s, f, g)$ to the case $m<m'$. If we exclude the case in which both m and m' are integers, there are three cases:

I. $m\notin \boldsymbol{Z}$ and $m'\in \boldsymbol{Z}$,
II. $m\in \boldsymbol{Z}$ and $m'\notin \boldsymbol{Z}$,
III. $m\notin \boldsymbol{Z}$ and $m'\notin \boldsymbol{Z}$.

The first case was treated by Sturm [**17**], [**18**]. Therefore we consider here the remaining two cases. Let $m'=k/2$ with an odd integer k, and assume that f

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