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Homomorphisms of Galois groups of solvably closed Galois extensions

By Kôji UCHIDA

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Let k_1 and k_2 be algebraic number fields of finite degrees. Let Ω_1 and Ω_2 be solvably closed Galois extensions of k_1 and k_2, respectively. Let $G_1=G(\Omega_1/k_1)$ and $G_2=G(\Omega_2/k_2)$ be their Galois groups. If G_1 and G_2 are isomorphic as topological groups, it is known that Ω_1 and Ω_2 are isomorphic fields, more precisely:

THEOREM [3]. *Let $\sigma: G_1\to G_2$ be an isomorphism of topological groups. Then there corresponds a unique isomorphism $\tau: \Omega_2\to\Omega_1$ such that $\tau\cdot\sigma(g_1)=g_1\tau$ for any $g_1\in G_1$.*

Looking at the statement above, it is natural to ask if the isomorphism σ can be replaced by a homomorphism.

CONJECTURE. *Let $\sigma: G_1\to G_2$ be a continuous homomorphism such that $\sigma(G_1)$ is open in G_2. Then there corresponds a unique injection $\tau: \Omega_2\to\Omega_1$ of fields such that $\tau\cdot\sigma(g_1)=g_1\tau$ for any $g_1\in G_1$.*

This conjecture means $\tau(\Omega_2)$ is G_1-invariant, $\tau(k_2)\subset k_1$ and $\Lambda_1=k_1\cdot\tau(\Omega_2)$ is a Galois extension of k_1 which corresponds to the kernel of σ. The Galois group $G(\Lambda_1/k_1)$ is isomorphic to an open subgroup of G_2. Then our conjecture may also be regarded as an extension of the theorem above to a non-solvably-closed extension Λ_1/k_1.

In the following, let k_1, k_2, Ω_1, Ω_2, G_1 and G_2 be as above, though we do not assume k_2 is of finite degree in the corollary of Theorem 2. Let $\sigma: G_1\to G_2$ be a homomorphism as in the conjecture, except in Theorem 2 where we do not assume $\sigma(G_1)$ is open. Let Λ_1 be the subfield of Ω_1 corresponding to the kernel of σ. Let E_2 be an extension of k_2 contained in Ω_2, and let U_2 be the corresponding subgroup of G_2. Let E_1 be the subfield of Ω_1 corresponding to $\sigma^{-1}(U_2)$. We call E_1 is the field corresponding to E_2 by σ.

1. Let $\mathfrak{p}_1$ be a finite prime of k_1. Let $G_{\mathfrak{p}_1}$ be a decomposition subgroup of $\mathfrak{p}_1$ in G_1. If $\sigma(G_{\mathfrak{p}_1})\neq(e)$ and if $\sigma(G_{\mathfrak{p}_1})$ is contained in some decomposition subgroup of a finite prime $\mathfrak{p}_2$ of k_2, $\mathfrak{p}_2$ is uniquely determined by $\mathfrak{p}_1$. Thus we get a mapping $\phi: \mathfrak{p}_1\mapsto\mathfrak{p}_2$ from a set of finite primes of k_1 into a set of finite primes of k_2. We will see below that almost all primes of k_2 are in the image of ϕ.