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Isometric immersions into spheres

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Recently ([1], [2]) new quantitative results concerning isometric immersions of complete Riemannian manifolds into euclidean space were obtained using a powerful theorem of Omori's [6]. Here we shall prove analogous results (theorem 1 and 2 below) concerning immersions into spheres. We begin with some auxiliary formulae for the sphere.

Let S_{λ}^{n+q} be the $(n+q)$-sphere of radius λ with the standard Riemannian metric induced by inclusion in R^{n+q+1}. For a point P_0 of S_{λ}^{n+q} say the north pole, and a nonnegative number h, let $C(P_0, h)$ be the $(n+q-1)$-hypersphere of S_{λ}^{n+q} with constant mean curvature h centered at P_0 and lying in the northern hemisphere. Note that $C(P_0, 0)$ is a great $(n+q-1)$-hypersphere in S_{λ}^{n+q}, the "equator". Let $D(P_0, h)$ be the closed geodesic ball around P_0 with $\partial D(P_0, h)=C(P_0, h)$. We take as origin of R^{n+q+1} the point P_0 and let φ be the position vector of a point in $D(P_0, h)$. If O is the center of the sphere S_{λ}^{n+q}, we set $e_0=\lambda^{-1}\overrightarrow{P_0O}$. If we denote by N the outer unit normal of S_{λ}^{n+q} in R^{n+q+1}, then by easy computations we obtain

$$d^2(P_0, C(P_0, h))=2\lambda^2[1-h\lambda(1+h^2\lambda^2)^{-1/2}], \tag{1}$$

where $d(P_0, C(P_0, h))$ is the distance in R^{n+q+1}, and

$$\lambda^{-1}\langle N, \varphi\rangle=\frac{1}{2}\lambda^{-2}\langle\varphi, \varphi\rangle\leqq\frac{1}{2}\lambda^{-2}d^2(P_0, C(P_0, h))=1-h\lambda(1+h^2\lambda^2)^{-1/2}, \tag{2}$$

where $\langle\,,\rangle$ is the standard inner product in R^{n+q+1}.
Also, for all unit vectors e which are tangent to S_{λ}^{n+q} at any point of $D(P_0, h)$ we have

$$|\langle e, e_0\rangle|\leqq(1+h^2\lambda^2)^{-1/2} \tag{3}$$

The proofs of the results in this paper will consist in simple applications of by Omori's theorem A in [6] which we now formulate.

THEOREM A. *Let M be a complete Riemannian manifold with sectional curvature bounded from below, consider a smooth function $f: M\to R$ with $\sup f<\infty$; then for any $\varepsilon>0$ there exists a point $p\in M$, which depends on ε, where $\|\mathrm{grad} f\|<\varepsilon$ and $\nabla^2 f(X, X)<\varepsilon$, for all unit vectors X of T_pM (by $\nabla^2 f$ we denote the*