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Completely positive maps in the tensor products of von Neumann algebras

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Let M_i and N_i be von Neumann algebras and τ_i be completely positive maps from M_i to N_i ($i=1, 2$). Then there exists a completely positive map $\tau_1 \otimes \tau_2$ (called the product map of τ_1 and τ_2) from the spatial C^*-tensor product $M_1 \otimes_\alpha M_2$ to the spatial C^*-tensor product $N_1 \otimes_\alpha N_2$ such that $\tau_1 \otimes \tau_2 (a \otimes b) = \tau_1(a) \otimes \tau_2(b)$. Moreover, when N_1 and N_2 are von Neumann subalgebras of M_1 and M_2 and both τ_1 and τ_2 are projections of norm one to N_1 and N_2 it is known ([**6**], [**9**]) that the map $\tau_1 \otimes \tau_2$ can further be extended, without the normality of τ_1 and τ_2, to the von Neumann tensor product $M_1 \bar{\otimes} M_2$ so that the resulting extension τ becomes a projection of norm one of $M_1 \bar{\otimes} M_2$ to the von Neumann subalgebra $N_1 \bar{\otimes} N_2$. This result is used to show a basic fact for injective von Neumann algebras; if M and N are injective von Neumann algebras, then $M \bar{\otimes} N$ is injective ([**6**], [**9**]).

In Section 2 we shall show that the completely positive extension of $\tau_1 \otimes \tau_2$ to the algebra $M_1 \bar{\otimes} M_2$ always exists for a couple of completely positive maps (τ_1, τ_2). In the proof, regarding the positive elements of $M_1 \bar{\otimes} M_2$ as completely positive maps from M_{1*} to M_2, we construct it.

Next in Section 3, we determine the normal part (σ-weakly continuous part) of the above extention in terms of the normal part of τ_1 and τ_2. The corresponding result for projections of norm one ([**11**; Theorem 3.1]) is used to classify certain types of maximal abelian subalgebras.

1. Preliminaries.

Let M and N be von Neumann algebras. We denote by M^* and M_*, the dual of M and the predual of M, and by $M \otimes N$ and $M \bar{\otimes} N$, the algebraic tensor product of M and N and the von Neumann tensor product of M and N respectively. For each $\phi \in M_*$ (resp. $\psi \in N_*$) we can define a σ-weakly continuous linear map R_ϕ of $M \bar{\otimes} N$ to N (resp. L_ψ of $M \bar{\otimes} N$ to M) which we call the right slice map (resp. left slice map) such that

$$R_\phi(a \otimes b) = \langle a, \phi \rangle b$$
$$(\text{resp. } L_\psi(a \otimes b) = \langle b, \psi \rangle a).$$