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Differentiability of solutions of some unilateral problem of parabolic type

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Let us begin with the following simple example of a parabolic unilateral problem

$$\left.\begin{array}{l}\partial u/\partial t-\Delta u\geqq 0\,, \qquad u\geqq \Psi \\ (\partial u/\partial t-\Delta u)(u-\Psi)=0\end{array}\right\} \quad \text{in} \quad \Omega\times(0, T] \tag{0.1}$$

$$u=0 \qquad \text{on} \quad \Gamma\times(0, T] \tag{0.2}$$

$$u(x, 0)=u_0(x)\geqq \Psi(x) \qquad \text{in} \quad \Omega\,. \tag{0.3}$$

Here Ω is a domain in R^N with sufficiently smooth boundary Γ, and Ψ is a function such that $\Psi\in W^{2,p}(\Omega)$ and $\Psi|_\Gamma\leqq 0$. We wish to make p small; however, assume

$$1<p<2<p^*=pN/(N-p)\,. \tag{0.4}$$

In view of Sobolev's imbedding theorem it follows that

$$W^{2,p}(\Omega)\subset H^1(\Omega)\subset L^{p'}(\Omega)\,, \qquad p'=p/(p-1)\,. \tag{0.5}$$

Let L_q be the realization of $-\Delta$ in $L^q(\Omega)$ under the Dirichlet boundary condition, and M_q be the multivalued mapping defined by

$$D(M_q)=\{u\in L^q(\Omega): u\geqq \Psi \text{ a.e. in } \Omega\}\,, \tag{0.6}$$

$$\begin{aligned} M_q u=\{g\in L^q(\Omega): g\leqq 0 \text{ a.e. in } \Omega,& \\ g(x)=0 \quad \text{if} \quad u(x)>\Psi(x)\}\,.& \end{aligned} \tag{0.7}$$

Note that $M_2=\partial I_K$ where I_K is the indicatrix of the closed convex set $K=D(M_2)$. The problem (0.1)-(0.3) is formulated in $L^p(\Omega)$ as

$$du(t)/dt+(L_p+M_p)u(t)\ni 0 \tag{0.8}$$

$$u(0)=u_0\,. \tag{0.9}$$

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