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Some remarks on lagrangian imbeddings

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§ 1. Introduction.

In this note we shall investigate some properties of lagrangian imbeddings of compact smooth n-manifolds into the complex n-space $\boldsymbol{C}^n$ from the view point of differential topology.

As for lagrangian immersions, considerably many facts are known. One of the most interesting results already obtained would be the theorem due to Gromov [3], Lees [5], Weinstein [11] which says that a smooth n-manifold M admits a lagrangian immersion into $\boldsymbol{C}^n$ if and only if the complexification $\tau(M)\otimes_{\boldsymbol{R}}\boldsymbol{C}$ of the tangent bundle $\tau(M)$ of M is a trivial complex vector bundle. On the other hand, Lees [5] obtained the homotopy theoretic classification theorem of lagrangian immersions of a smooth n-manifold L into a smooth symplectic $2n$-manifold M.

On the contrary, as for lagrangian imbeddings very few are known. First we consider what kind of compact manifolds admit lagrangian imbeddings into $\boldsymbol{C}^n$. A familiar example of a compact manifold admitting lagrangian imbeddings into $\boldsymbol{C}^n$ is the n-torus T^n whose lagrangian imbeddings are defined by n functions in involution (see [2]). However we can prove the following theorem which shows that there are many other examples of compact orientable manifolds than T^n which admit lagrangian imbeddings into $\boldsymbol{C}^n$ if $n\geqq 3$. (For the case $n=2$, it is easily seen that T^2 is the only compact orientable surface that admits a lagrangian imbedding into $\boldsymbol{C}^2$.)

THEOREM 1. *Let M be a compact orientable smooth n-manifold which admits an immersion into the euclidean $(n+1)$-space $\boldsymbol{R}^{n+1}$. Then $M\times S^1$ admits a lagrangian imbedding into $\boldsymbol{C}^{n+1}$.*

Since an orientable smooth n-manifold M admits an immersion into $\boldsymbol{R}^{n+1}$ if and only if it is s-parallelizable, Theorem 1 can be restated as follows.

COROLLARY 1. *Let M be a compact smooth s-parallelizable n-manifold. Then $M\times S^1$ admits a lagrangian imbedding into $\boldsymbol{C}^{n+1}$.*

Recalling that every compact n-manifold M with $n\leqq 3$ is immersible in $\boldsymbol{R}^{n+1}$ (for $n=3$, see [4]), we have:

COROLLARY 2. *Let M be a compact orientable smooth n-manifold with $n\leqq 3$.*