On defining equations of symmetric submanifolds in complex projective spaces

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Introduction.

Let M be a compact complex manifold and L a holomorphic line bundle on M. Let $\Gamma(L)$ denote the vector space of all holomorphic sections of L and let $P(\Gamma(L))$ denote the projective space of hyperplanes of $\Gamma(L)$. A holomorphic line bundle L on M is said to be very ample if we can define a map $j_L : M \to P(\Gamma(L))$ by $j_L(x) = \{s \in \Gamma(L) \mid s(x) = 0\}$ for $x \in M$ and furthermore j_L is a holomorphic imbedding.

A compact simply connected homogeneous complex manifold M is called a C-space. If M has a Kähler metric it is said to be a kählerian C-space. Let L be a very ample holomorphic line bundle on a kählerian C-space M. Consider the homogeneous ideal of the projective submanifold $j_L(M)$ in $P(\Gamma(L))$. For example, for a complex Grassmann manifold M imbedded into a projective space by the Plücker coordinates, it is known that the homogeneous ideal of M is generated by quadrics. Moreover E. Cartan has realized in his Thèse some exceptional complex simple Lie groups as the projective automorphism groups of projective submanifolds defined by some quadrics — these projective submanifolds are all kählerian C-spaces. (See [4] pp. 272-276.)

Motivated by these facts, we ask whether the homogeneous ideal of $j_L(M)$ of a kählerian C-space M is generated by quadrics or not. In this note we shall prove that if M is a Hermitian symmetric space of compact type the answer is affirmative for each L (Corollary of Main Theorem). We give also a sufficient condition for a general kählerian C-space in order that the question is affirmative (Main Theorem).

For a compact projective manifold M and a very ample holomorphic line bundle L on M, Mumford [7] has given a cohomological condition in order that the homogeneous ideal of $j_L(M)$ is generated by quadrics. Our basic formulation in section 1 is due to Mumford [7], while our condition for kählerian C-spaces is not for the cohomologies of L but for the Chern class of L.

After having finished this work, the authors learned that our Corollary to