Regular embeddings of $C^*$-algebras in monotone complete $C^*$-algebras

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Introduction.

Let $A$ be a unital $C^*$-algebra and $A_{s.a.}$ the self-adjoint part of $A$. If each bounded increasing net (resp. sequence) in $A_{s.a.}$ has a supremum then $A$ is said to be monotone (resp. monotone $\sigma$-) complete. [In the literature, e.g., [10, 16, 20], the adjective “monotone (resp. monotone $\sigma$-) complete” is employed as a synonym for “monotone (resp. monotone $\sigma$-) complete”, but in this paper we will use it in a different sense (cf. Definition 1.2).] As was shown by J. D. M. Wright [22], each unital $C^*$-algebra $A$ possesses a unique regular $\sigma$-completion, i.e., a monotone $\sigma$-complete $C^*$-algebra $\hat{A}$ which contains $A$ as a $C^*$-subalgebra and satisfies the following properties:

i) $\hat{A}_{s.a.}$ itself is a unique monotone $\sigma$-closed subspace of $\hat{A}_{s.a.}$ which contains $A_{s.a.}$;

ii) each $x$ in $\hat{A}_{s.a.}$ is the supremum in $\hat{A}_{s.a.}$ of \{a $\in A_{s.a.}$; $a \leq x$\}; and

iii) whenever a subset $\mathcal{F}$ of $A_{s.a.}$ has a supremum $x$ in $A_{s.a.}$, then $x$ remains the supremum of $\mathcal{F}$ in $\hat{A}_{s.a.}$.

On the other hand the present author proved in [6] that each unital $C^*$-algebra $A$ has a unique injective envelope, which will be written as $I(A)$, i.e., a minimal injective $C^*$-algebra containing $A$ as a $C^*$-subalgebra. In this paper we give a monotone complete version of the above J. D. M. Wright’s result by embedding $A$ in its injective envelope $I(A)$ (Theorem 3.1). Namely it is shown that the monotone closure $\overline{A}$ of $A$ in $I(A)$ is a monotone complete $C^*$-algebra which satisfies the above properties i), ii) and iii) with $\hat{A}$ replaced by $\overline{A}$ and moreover “monotone $\sigma$-” in i) replaced by “monotone”. We call $\overline{A}$ the regular monotone completion of $A$. To see that $\overline{A}$ satisfies ii) we consider the family of all unital $C^*$-algebras which contain $A$ as a $C^*$-subalgebra and satisfy ii) (called “regular extensions” of $A$) and we show that, instead of $\overline{A}$, a maximal regular extension of $A$, written $\tilde{A}$, is realized as a monotone closed $C^*$-subalgebra of $I(A)$, hence that $\overline{A} \subset \tilde{A}$ satisfies ii). By the construction we have the canonical inclusions $A \subset \hat{A} \subset \overline{A} \subset \tilde{A} \subset I(A)$; however it remains open.