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On the units of the integral group ring of a dihedral group

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0. Introduction.

For G an arbitrary finite group, ZG denotes the integral group ring and $U(ZG)$ its group of units. We denote by ε the augmentation from ZG to Z and by $V(ZG)$ the subgroup of units u of ZG with $\varepsilon(u)=1$; clearly $U(ZG)=V(ZG)\times U(Z)$. In this paper we study $U(ZD_n)$ where D_n is a dihedral group of order $2n$. Throughout this paper we assume that n is an odd integer and all modules are finitely generated left modules. Main results in this paper are the following;

THEOREM A. *$V(ZD_n)$ is a semi-direct product of a torsion free normal subgroup with D_n.*

THEOREM B. *There are $\phi(n)/2$ conjugate classes in $V(ZD_n)$ of subgroups of $V(ZD_n)$ isomorphic to D_n if the order of the locally free class group $C(ZD_n)$ of ZD_n is odd. Here ϕ denotes Euler's totient function.*

By [3] $D(ZD_n)=0$ if $n<60$. Masley's results in [5] show that values of n satisfying the condition of Theorem B and less than 60 are 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 31, 33, 35, 39, 45, 51, 55 and 57. It seems to be an interesting problem to delete the condition on $C(ZD_n)$ in Theorem B.

Let D_n be generated by σ and τ with relations $\sigma^n=\tau^2=1$ and $\tau^{-1}\sigma\tau=\sigma^{-1}$. Set $S=ZD_n/(1+\sigma+\sigma^2+\cdots+\sigma^{n-1})$. The key point in proving Theorems A and B is that the order S behaves like a hereditary order as far as locally S-modules concern. For example the locally free class group of S is isomorphic to that of the center of S. For other applications of this property of S, see [6].

For $n=3$ complete results are obtained by Hughes and Pearson [4]. Further information on $V(ZD_3)$ and especially on the torsion free normal subgroup in Theorem A is found in the excellent survey article on the unit group of rings by Dennis [2].

Recently K. Sekiguchi (Tokyo Metropolitan University) has extended Theorem A to a metabelian group G such that the exponent of G/G' is 1, 2, 3, 4 or 6, where G' denotes the commutator subgroup of G.