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On the relative Mordell-Weil rank of elliptic quartic curves

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Let A be an abelian variety defined over a number field k of finite degree over the rationals $\boldsymbol{Q}$. For a finite extension K of k, let A_K be the group of points of A rational over K. As is well-known, the group A_K is finitely generated [L]. For any finitely generated abelian group G, let $\mathrm{rk}(G)$ be the rank of G. We put $\rho_K(A)=\mathrm{rk}(A_K)$, the Mordell-Weil rank of A with respect to K. By the relative Mordell-Weil rank of A with respect to K/k, we shall mean the difference $\rho_{K/k}(A)=\rho_K(A)-\rho_k(A)$.

In this paper, we shall study this quantity when A is an elliptic quartic curve and K/k is a quadratic extension. Among elliptic curves under consideration, the curve $E(\kappa)$ for non-zero $\kappa\in k$ defined by equations

$$\begin{cases} X_0^2+\kappa X_1^2=X_2^2\,, \\ X_0^2-\kappa X_1^2=X_3^2 \end{cases}$$

has multiple interests. For example, we shall show that

$$\rho_{k(\sqrt{\lambda})/k}(E(\kappa))=\rho_{k(\sqrt{\kappa})/k}(E(\lambda))$$

whenever κ, λ are non-square elements of k. Next, let $k=\boldsymbol{Q}$, and let κ be a square free natural number. Then we shall obtain the relations

$$\rho_K(E(\kappa))=\rho_{\boldsymbol{Q}}(E(\kappa)) \quad \text{when} \quad K=\boldsymbol{Q}(\sqrt{\kappa}) \quad \text{or} \quad \boldsymbol{Q}(\sqrt{-\kappa})\,,$$

$$\rho_K(E(\kappa))=2\rho_{\boldsymbol{Q}}(E(\kappa)) \quad \text{when} \quad K=\boldsymbol{Q}(\sqrt{-1})\,.$$

In the Appendix, I have collected miscellaneous facts and comments on the (absolute) Mordell-Weil rank $\rho_{\boldsymbol{Q}}(\kappa)$ of $E(\kappa)$ where κ is a square free natural number.

1. We begin with a single lemma on any abelian variety. Let A be an abelian variety defined over a number field k. Assume that K/k is a finite galois extension with the galois group G. We then consider the homomorphism $T_{K/k}\colon A_K\to A_k$ defined by $T_{K/k}(x)=\sum_{\sigma\in G}x^\sigma$, the trace.

(1.1) LEMMA. $\rho_{K/k}(A)=\mathrm{rk}(\mathrm{Ker}\, T_{K/k})$.