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Minimal length of Liouville chain for solutions of an algebraic differential equation

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§ 0. Introduction.

The author [5] proved that the order of a liouvillian element in Liouville's sense is at most 3 if it satisfies an algebraic differential equation of the first order. Here, we shall generalize his theorem as follows: The order of a liouvillian element in Liouville's sense is at most $3n$ if it satisfies an algebraic differential equation of order n.

Let k be an ordinary differential field of characteristic 0, and Ω be a universal extension of k. We assume that the field of constants k_0 of k is algebraically closed. A finite chain of extending differential subfields $L_0 \subset L_1 \subset \cdots \subset L_n$ in Ω is called a *Liouville chain* over k if the following three conditions are satisfied:

(i) L_0 is an algebraic extension of k of finite degree:

(ii) The field of constants of L_n is k_0:

(iii) For each i $(1 \leqq i \leqq n)$ there exists a finite system of elements $w_1, \cdots, w_r$ of L_i which satisfies the following two conditions; either $w'_j \in L_{i-1}$ or w'_j/w_j is the derivative of an element of L_{i-1} for each j $(1 \leqq j \leqq r)$; L_i is an algebraic extension of $L_{i-1}(w_1, \cdots, w_r)$ of finite degree.

A subfield L of Ω is called a *liouvillian extension* of k if there exists a Liouville chain over k which ends with L. Let z be an element of Ω. Then, z is called a *liouvillian element* over k if there exists a Liouville chain over k such that its end contains z. In particular, if $k=k_0(x)$ with $x'=1$, then a liouvillian element over k is called an *elementary transcendental function* of x over k_0 (cf. Watson [9, p. 111]). The following definition is due to Liouville [3]: A liouvillian element z over k is said to be of *order* m if m is the minimum of those n such that the end of a Liouville chain $L_0 \subset \cdots \subset L_n$ over k contains z.

THEOREM. *The order of a liouvillian element over k satisfying an algebraic differential equation over k of order n is at most $3n$.*

It follows from the following:

LEMMA. *Let k^* be a finitely generated differential extension field of k in Ω*