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Classes on ZF models

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Let $\mathcal{A}=(A, E)$ be a model of ZF where A is a set and $E\subseteqq A\times A$, and $\bar{K}$ a new predicate letter. We say that a subset K of A is a class of $\mathcal{A}$ if and only if $[\mathcal{A}, K]$ is a model of $ZF(\bar{K})$ where $\bar{K}$ is interpreted by K and the replacement scheme holds for all formulae involving both $\in$ and the new predicate letter $\bar{K}$. In this paper we prove some results about classes.

A class K of $\mathcal{A}$ is definable if and only if for some formula $\phi(v_0, v_1, \cdots, v_n)$ not involving $\bar{K}$ and some elements $a_1, a_2, \cdots, a_n$ of A, $K=\{x\in A\mid \mathcal{A}\models\phi(x, a_1, a_2, \cdots, a_n)\}$. We denote by $\mathrm{def}(\mathcal{A})$ the set of all definable classes of $\mathcal{A}$, and say that a class K of $\mathcal{A}$ is undefinable if and only if $K\notin \mathrm{def}(\mathcal{A})$. Let κ be a strongly inaccessible cardinal. Then V_κ is a model of ZF and every subset of V_κ is a class of V_κ. Since $|\mathrm{def}(V_\kappa)|=|V_\kappa|<2^{|V_\kappa|}$, there exist undefinable classes of V_κ. In section 1, we prove the following:

THEOREM. *If $\mathcal{A}$ is a standard model of ZF, then there exists an undefinable class of $\mathcal{A}$.*

If $\mathcal{A}$ is a model of ZF, then $[\mathrm{def}(\mathcal{A}), A]$ is a model of GB (Gödel Bernays set theory). Theorem means that if $\mathcal{A}$ is standard, then there exists $N\supsetneqq \mathrm{def}(\mathcal{A})$ such that $[N, A]$ is a model of GB.

Let K and K' be classes of $\mathcal{A}$. K and K' are incompatible if and only if $[\mathcal{A}, K, K']\not\models ZF(\bar{K}, \bar{K}')$ where $\bar{K}$ and $\bar{K}'$ are new predicate letters and $ZF(\bar{K}, \bar{K}')$ are axioms of ZF in the language $(\in, \bar{K}, \bar{K}')$. There are many incompatible classes in countable models of ZF (Mostowski [7]). The existence of incompatible classes means that $ZF(\bar{K}, \bar{K}')$ and $ZF(\bar{K})+ZF(\bar{K}')$ are not equivalent, in other words, there exists a sentence Φ such that $ZF(\bar{K}, \bar{K}')\vdash\Phi$ but $ZF(\bar{K})+ZF(\bar{K}')\nvdash\Phi$. In section 2, we present such a sentence Φ explicitly under some assumption.

1. Undefinable classes.

We begin with some definitions from model theory. Let $\mathcal{L}$ be a first order language and P a class of structures of $\mathcal{L}$. P is inductive if and only if the union of any chain $M_0\subseteqq M_1\subseteqq \cdots \subseteqq M_\alpha\subseteqq \cdots (\alpha<\lambda)$ of structures from P is again in P. Let $\phi(v_1, v_2, \cdots, v_n)$ be a formula of $\mathcal{L}$. ϕ is said to be P-persistent