J. Math. Soc. Japan
Vol. 32, No. 3, 1980

Two local ergodic theorems on L_∞

Dedicated to Professor Shisanji Hokari on his 70th birthday

By Ryotaro SATO

(Received April 28, 1978)

Introduction.

Let $(X, \mathfrak{F}, \mu)$ be a σ-finite measure space and let $L_p(X)=L_p(X, \mathfrak{F}, \mu)$, $1 \leqq p \leqq \infty$, be the usual Banach spaces of real or complex functions on $(X, \mathfrak{F}, \mu)$. In this paper we first prove that if $(T_t: t>0)$ is a strongly continuous one-parameter semigroup of linear contractions on $L_1(X)$ such that $\mu(A)>0$ implies $\int_A |T_t f| d\mu>0$ for some $f \in L_1(X)$ and $t>0$, then the local ergodic theorem holds for the adjoint semigroup $(T_t^*: t>0)$ acting on $L_\infty(X)$, i.e. for any $f \in L_\infty(X)$ there exists a scalar function $T_t^* f(x)$, measurable with respect to the product of the Lebesgue measurable subsets of $(0, \infty)$ and $\mathfrak{F}$, such that for each fixed $t>0$, $T_t^* f(x)$ as a function of x belongs to the equivalence class of $T_t^* f$, and the following local ergodic limit

$$\lim_{b \to 0} \frac{1}{b} \int_0^b T_t^* f(x) dt$$

exists and is finite a.e. on X; in particular, $\lim_{t \to 0} \|T_t v - v\|_1 = 0$ for all $v \in L_1(X)$ if and only if

$$\lim_{b \to 0} \frac{1}{b} \int_0^b T_t^* f(x) dt = f(x) \quad \text{a.e.}$$

for all $f \in L_\infty(X)$. This generalizes Krengel's local ergodic theorem [7] for semigroups of nonsingular point transformations on $(X, \mathfrak{F}, \mu)$. For another related result we refer the reader to the author [13], in which positive semigroups on $L_1(X)$ are considered and a similar result is obtained, only assuming that the positive semigroup $(T_t: t>0)$ is strongly integrable over every finite interval. We next prove that if $(T_t: t>0)$ is a strongly continuous one-parameter semigroup of bounded linear operators on $L_p(X)$ for some fixed p, $1 \leqq p < \infty$, such that $(T_t: t>0)$ is also simultaneously a semigroup of linear contractions on $L_\infty(X)$ with $T_t^* L_1(X) \subset L_1(X)$ for all $t>0$, then under one of the following two conditions (I) and (II), the local ergodic theorem holds for $(T_t: t>0)$ on