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Cartan subgroups of a Lie group

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§ 1. Introduction.

By an analytic group and by an analytic subgroup of a Lie group, we mean a connected Lie group and a connected Lie subgroup, respectively. An analytic subgroup and the corresponding Lie algebra will be denoted by the same capital script and capital Roman letter, respectively. For example, if $\mathcal{L}$ is an analytic group and $\mathcal{G}$ an analytic subgroup of $\mathcal{L}$, then L will denote the Lie algebra of $\mathcal{L}$ and G the subalgebra of L corresponding to $\mathcal{G}$.

Let $\mathcal{G}$ be an analytic group, and H a Cartan subalgebra of G. We shall call $\mathcal{H}$ a *Cartan subgroup* of $\mathcal{G}$. Any Cartan subgroup of $\mathcal{G}$ is closed in $\mathcal{G}$. For the closure of a subset $\mathcal{M}$ of a topological space, we adopt the notation $\overline{\mathcal{M}}$. In this paper, we shall prove the following theorems:

THEOREM 5. *Let $\mathcal{L}$ be an analytic group, and $\mathcal{G}$ an analytic subgroup of $\mathcal{L}$. Then there exists a Cartan subgroup $\mathcal{H}$ of $\mathcal{G}$ such that $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$. This implies, in particular, if $\mathcal{G}$ is non-closed, then so is $\mathcal{H}$.*

THEOREM 6. *Let $\mathcal{G}$ be an analytic subgroup of $GL(n, \boldsymbol{R})$. For any Cartan subgroup $\mathcal{H}$ of $\mathcal{G}$ we have $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$.*

REMARK. In Theorem 5 and 6, we can replace $\bar{\mathcal{G}}=\overline{\mathcal{H}}\mathcal{G}$ by $\bar{\mathcal{G}}=\overline{Z(\mathcal{H})}\mathcal{G}$ where $Z(\mathcal{H})$ is the center of $\mathcal{H}$. Indeed, since $\mathcal{H}$ is nilpotent, we have that $\overline{\mathcal{H}}=\overline{Z(\mathcal{H})}\mathcal{H}$.

Let L be a Lie algebra. A Cartan subalgebra of L is, by definition, the eigenspace corresponding to 0 of any regular inner derivation of L. Also we know that a subalgebra H of L is Cartan if and only if H is nilpotent and coincides with its normalizer.

In [2], Gantmacher proved that for complex semisimple Lie algebras we can define Cartan subalgebras using inner automorphisms rather than derivations. We shall generalize the result of Gantmacher, and get a new definition of Cartan subalgebras. Let $\mathcal{L}$ be an analytic group. For $a\in\mathcal{L}$, we denote by $\mathrm{Ad}\,a$ the (inner) automorphism of L induced by the inner automorphism $x\mapsto axa^{-1}$ of the group $\mathcal{L}$.

THEOREM 1. *Let $\mathcal{L}$ be an analytic group, and let x be a regular element of $\mathcal{L}$, i.e., the multiplicity of the eigenvalue one of* $\mathrm{Ad}\,x$ *is the smallest. Let H be the eigenspace corresponding to the eigenvalue one of* $\mathrm{Ad}\,x$.