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Weakly closed dihedral 2-subgroups in finite groups

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1. Introduction.

Let S be a Sylow 2-subgroup of a finite group G. A subgroup T of S is said to be weakly closed in S with respect to G if T has the following property; "Whenever $T\subseteqq S$, $g\in G$, and $T^g\subseteqq S$, then $T^g=T$." Let φ be the natural homomorphism from G onto $G/O(G)$. Then we set $Z^*(G)=\varphi^{-1}(Z(G/O(G)))$. The object of this paper is to prove the following results.

THEOREM I. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a dihedral subgroup T of S is weakly closed in S (with respect to G). Then one of the following holds;*

(i) $T\nsubseteqq\langle Z(T)^G\rangle$,

(ii) $Z(T)\subseteqq Z^*(G)$,

(iii) $|T|=4$, *and Sylow 2-subgroups of* $\langle T^G\rangle$ *are T or dihedral of order 8 or* $\langle T^G\rangle/O(\langle T^G\rangle)\cong U_3(4)$,

(iv) $|T|=8$, *and Sylow 2-subgroups of* $\langle T^G\rangle$ *are dihedral or semi-dihedral,*

(v) $|T|\geqq 16$, *and Sylow 2-subgroups of* $\langle Z(T)^G\rangle$ *are dihedral or semi-dihedral or are wreath products.*

THEOREM II. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a generalized quaternion subgroup Q of S is weakly closed in S. Let $\langle z\rangle=Z(Q)$. Then one of the following holds;*

(i) $Q\nsubseteqq\langle z^G\rangle$,

(ii) $z\in Z^*(G)$,

(iii) $\langle z^G\rangle/O(\langle z^G\rangle)$ *is* M_{11}, M_{12}, $\hat{M}_{12}$, $L_3(q)$, $U_3(q)$, $G_2(q)$, *or* ${}^3D_4(q)$, *q odd.*

THEOREM III. *Let S be a Sylow 2-subgroup of a finite group G. Suppose a semi-dihedral subgroup D of S is weakly closed in S. Then $Z(D)\subseteqq Z^*(G)$ or Sylow 2-subgroups of $\langle Z(D)^G\rangle$ are dihedral or semi-dihedral.*

2. Preliminaries.

LEMMA 2.1. *Suppose a subgroup A of a Sylow p-subgroup P of G is conjugate to a normal subgroup B of P in G. Then there exists an element $g\in G$ such that $A^g=B$ and $N_P(A)^g\subseteqq P$.*