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Remarks on a geometric constant of Yau

By David HOFFMAN

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§ 1. Introduction.

Let M be a compact m-dimensional Riemannian manifold with or without boundary. In [2], Yau defines an isoperimetric constant $I(M)$ as follows: $I(M)=\inf\frac{\mathrm{Vol}(\partial M_1\cap\partial M_2)}{\min(\mathrm{Vol}\,M_1,\ \mathrm{Vol}\,M_2)}$, the infimum being taken over all decompositions $M=M_1\cup M_2$ with $\mathrm{Vol}(M_1\cap M_2)=0$. By standard methods, Yau shows that

$$I(M)=\inf\left\{\int_M|\nabla f|\Big/\inf_{\beta\in R}\int_M|f-\beta|\ \Big|\ f\in C^1(M)\right\}. \tag{1}$$

$I(M)$ is useful for estimating eigenvalues of the Laplacian from below. In this note we wish to investigate and clarify a geometric quantity ω associated to M (and defined below) that arises in trying to estimate $I(M)$. At each $p\in M$, consider a subset $\mathcal{R}$ of T_p^1M with the following property: The set of all points of M reachable by minimal geodesics from p with initial direction in $\mathcal{R}$ has volume equal to or greater than $\mathrm{Vol}M/2$. Then ω_p is equal to the infimum of the $(m-1)$-dimensional areas of all such $\mathcal{R}$ and $\omega=\inf_{p\in M}\omega_p$. Let $\alpha_{m-1}=(\mathrm{m}-1)$-dimensional area of $S^{m-1}\subset \boldsymbol{R}^m$. Clearly $0<\omega/\alpha_{m-1}\leqq 1/2$ and for $M=S^m$, $\omega=\omega_p=\alpha_{m-1}/2$. One estimate we make is contained in the following proposition.

PROPOSITON 1. *Suppose the Ricci curvature of M is equal to or greater than $(m-1)a^2$. Then for all $p\in M$*

$$\omega_p/\alpha_{m-1}\geqq\omega/\alpha_{m-1}\geqq(1/2)V(a,\ d(M))^{-1}\cdot\mathrm{Vol}(M).$$

Here $V(a, \rho)$ is the volume of the solid ball of radius ρ in the space form of constant curvature a^2, (a may be a real positive or a purely imaginary number) and $d(M)$ is the diameter of M. The proof of Proposition 1 follows in § 3.

§ 2. To begin with, for $p\in M$ let (r, θ) denote polar coordinates on T_pM; $\theta\in T_p^1M$, $r\geqq 0$. For each $\theta\in T_p^1M$, let $r(\theta)$ be the distance to the cut locus of

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