A counterexample to a conjecture of Whitehead and Volodin-Kuznetsov-Fomenko

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In the study of 3-manifolds, to construct an algorithm of recognizing the standard 3-sphere $S^3$ among all 3-manifolds is a very important problem. The first basic work of this problem was done by Whitehead in 1936 [6], who discovered that certain (but not all) Heegaard diagrams for $S^3$ had a rather special geometric property (see Conjecture A in the paper). Later Volodin-Kuznetsov-Fomenko conjectured that Heegaard diagrams for $S^3$ are reducible except for the canonical one. But Birman states in [2] that “nobody has succeeded in verifying such an assertion between 1935 and 1977, or producing a counterexample”. Most recently Homma-Ochiai-Takahashi [3] proved that the conjecture is really true for the case of genus two. But in this paper we give a counterexample for the case of genus four. The Volodin-Kuznetsov-Fomenko-Whitehead algorithm is closely related with the algorithm to determine whether a knot is trivial or not and so our counterexample is constructed as a branched covering space over a trivial 5-bridge knot.

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1. Reducible Heegaard diagrams.

Let $M$ be a closed orientable 3-manifold and $W_1, W_2$ solid tori of genus $n$ and $h : \partial W_2 \rightarrow \partial W_1$ a homeomorphism of the boundary surfaces. Then the triple $(W_1, W_2; h)$ is called a Heegaard splitting of genus $n$ for $M$ when $M=W_1 \cup_h W_2$.

A properly embedded disk $D$ in a solid torus $W$ of genus $n$ is called a meridian-disk of $W$ if $cl(W-N(D, W))$ is a solid torus of genus $n-1$, and a collection of mutually disjoint $n$ meridian-disks $D_1, \ldots, D_n$ in $W$ is called a complete system of meridian-disks of $W$ if $cl(W- \bigcup_{i=1}^{n} N(D_i, W))$ is a 3-ball. We call a collection of mutually disjoint $(n+1)$ meridian-disks in $W$ an extended complete system of meridian-disks of $W$ provided that any $n$ subcollection is a complete system of meridian-disks of $W$. 