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# Quasi-maximal ideals and quasi-primary ideals of weak-*Dirichlet algebras

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Let $A$ be a weak-*Dirichlet algebra of $L^\infty(m)$ and let $H^\infty(m)$ denote the weak-*closure of $A$ in $L^\infty(m)$. Let $B$ be a weak-*closed subalgebra of $L^\infty(m)$ which contains $A$ and let $I_B=\{f\in B\,;\int fg\,dm=0\ \ g\in B\}$ and let $E^{\mathcal{B}}$ be a conditional expectation for $B\cap\bar{B}$. When $B=H^\infty(m)$, $I_B$ is a maximal ideal and $E^{\mathcal{B}}$ is multiplicative on $B$. When $B\neq H^\infty$, it is not known whether $E^{\mathcal{B}}$ is always multiplicative on $B$. It is easy to show that if $E^{\mathcal{B}}$ is multiplicative on $B$, $I_B$ is a quasi-maximal ideal of $B$ (Definition 2, § 4). It is known ([3]) that when $E^{\mathcal{B}}$ is multiplicative on $B$, if $M$ is a left continuous invariant subspace for $B$ of $L^\infty(m)$, then $M=\chi_E qB$ for some unimodular $q$ and some characteristic function $\chi_E$ in $B$. We show in this paper that the weak converse is valid, i.e., if any left continuous invariant subspace for $B$ of $L^\infty(m)$ has the form $\chi_E qB$, then $I_B$ is a quasi-maximal ideal of $B$. Secondly we show that if $I$ is the weak-*closed linear span of functions in $H^\infty(m)$, vanishing on sets of positive measure, then it is a primary ideal of $H^\infty(m)$. When $B\neq H^\infty(m)$, there exist quasi-primary ideals of $B$ (Definition 1, § 3). Thirdly we give the necessary and sufficient conditions for a minimum weak-*closed subalgebra of $L^\infty(m)$ that contains $H^\infty(m)$ properly. And we show that there exists at least one function in $H^\infty(m)$ that is not a weak-*limit of functions, vanishing on sets of positive measure if and only if there exists a minimum weak-*closed subalgebra of $L^\infty(m)$ that contains $H^\infty(m)$ properly.

## 1. Preliminaries.

Recall that by definition [5] a weak-*Dirichlet algebra, is an algebra of essentially bounded measurable functions on a probability measure space $(X, \mathcal{A}, m)$ such that (i) the constant functions lie in $A$; (ii) $A+\bar{A}$ is weak-*dense in $L^\infty=L^\infty(m)$ (the bar denotes conjugation); (iii) for all $f$ and $g$ in $A$, $\int_X fg\,dm=\int_X f\,dm\int_X g\,dm$. The abstract Hardy space $H^p=H^p(m)$, $1\leqq p\leqq\infty$,

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