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Variations of metrics on homogeneous manifolds

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0. Introduction.

In [5, p. 115] Wu-Yi Hsiang conjectured the following: Of all possible Riemannian metrics on a homogeneous manifold $M=K/H$ (K compact, semi-simple), the *natural* metric, corresponding to the Cartan-Killing form of the Lie algebra of K, should admit the largest isometry group. In [1] he tested this conjecture with the second Stiefel manifold $V^{n,2}=SO(n)/SO(n-2)$, $n>20$, odd. He claimed that $\dim \mathrm{ISO}(g)\leqq\frac{1}{2}n(n-1)+1$ for all Riemannian metrics g on $V^{n,2}$, where $\mathrm{ISO}(g)$ denotes the isometry group of g, and that equality holds only when g is the natural metric. However, in this paper we will establish the following:

THEOREM. *The second Stiefel manifold* $V^{n,2}$, $n\geqq 31$, *odd, has uncountably many homothetically distinct homogeneous metrics* g, *for which* $\dim \mathrm{ISO}(g)=\frac{1}{2}n(n-1)+1$. *Note that* $\dim V^{n,2}=2n-3$.

The procedure will be to study the space of K-invariant metrics on K/H and by explicit computation of sectional curvature, distinguish different metrics by homothety type. For terminology, see Section 1.

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1. Background material.

In this section we collect some results on the geometry of homogeneous spaces, all of which may be found in [3, Chapter X]. We study homogeneous manifolds $M=K/H$, where K acts as isometries of some Riemannian metric on M, hence also as a group of automorphisms of the principal $O(m)$-bundle over M associated with the metric. Since H is compact, M is *reductive*; that is, the Lie algebra $\mathfrak{k}$ of K admits a vector space decomposition

$$\mathfrak{k}=\mathfrak{h}+\mathfrak{m}$$

where $\mathfrak{h}$ is the Lie algebra of H, $\mathfrak{h}\cap\mathfrak{m}=0$, and $\mathrm{ad}(H)\mathfrak{m}\subseteqq\mathfrak{m}$.