Automorphic forms and the periods of abelian varieties

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(Received July 11, 1978)

There are three interrelated topics to be treated in this paper:

I. Monomial relations between the periods of abelian varieties with complex multiplication;

II. The derivatives of automorphic forms of arithmetic type;

III. The non-vanishing of the first cohomology group of a discrete subgroup of $SU(n, 1)$.

To describe our results, let $A$ be an abelian variety of dimension $g$ defined over $\mathbf{Q}$, whose endomorphism-algebra contains a totally imaginary quadratic extension $K$ of a totally real algebraic number field $F$ such that $[F: \mathbf{Q}]=g$. Throughout the paper, we denote by $\mathbf{Q}$ the algebraic closure of the rational number field $\mathbf{Q}$ embedded in the complex number field $C$. Let $\Phi$ be the representation of $K$ on the space of holomorphic 1-forms on $A$. Then $\Phi$ consists of $g$ injections $\tau$ of $K$ into $C$, and for each $\tau$, there is a $\mathbf{Q}$-rational 1-form $\omega_\tau$ on $A$ on which an element $x$ of $K$ acts as a scalar $x^\tau$. As shown in [13], there is a constant $p(\tau, \Phi)$ such that

$$\int_c \omega_\tau \sim \pi \cdot p(\tau, \Phi) \quad \text{for all 1-cycles } c \text{ on } A. \tag{0.1}$$

Here and henceforth, we write $a \sim b$ for two complex numbers $a$ and $b$ if $a/b \in \mathbf{Q}$. The first principal aim of this paper is to prove monomial relations between $p(\tau, \Phi)$ for various $\Phi$ with the same $K$, as well as relations between such “periods” for a given $K$ and those for an extension of $K$. The constant $p(\tau, \Phi)$ can be obtained from a Hilbert modular function $f$ with respect to a congruence subgroup of $GL_2(F)$ as follows. Take the variable $z=(z_1, \cdots, z_g)$ on the product $\mathfrak{H}_1^g$ of $g$ copies of the upper half plane $\mathfrak{H}_1$. Let $\Phi=\sum_{\nu=1}^g \tau_\nu$ and $w_\nu=(w^\tau_1, \cdots, w^\tau_g)$ with an element $w$ of $K$ such that $\text{Im}(w^\tau_\nu)>0$ for all $\nu$. Then

$$\left(\frac{\partial f}{\partial z_\nu}(w)\right) \sim \pi \cdot p(\tau_\nu, \Phi)^2 \quad (\nu=1, \cdots, g), \tag{0.2}$$

if $f$ is $\mathbf{Q}$-rational, i.e., if $f$ is the quotient of two Hilbert modular forms with

* Supported by NSF Grant MCS76-11376.