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Abstract aspects of asymptotic analysis

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Introduction

In the present paper we offer a formal treatment of some simplest classes of the asymptotic methods. Our result is summarized in Theorem 4.5 below. It tells when a given element admits an asymptotic expansion, and also shows the canonical way to derive its expansion.

In many branches of mathematics, various asymptotic methods provide powerful tools, often exhibiting a strong resemblance. This leads one to a suspicion that there be a common structure in these methods of analysis. For instance, in many classes of asymptotic analysis, an asymptotic expansion is just one into homogeneous parts, as a formal series expansion. Thus, for such classes, a speculation may be done that there be an action of the multiplicative group $\boldsymbol{R}_+$ of positive real numbers. We actually observe such $\boldsymbol{R}_+$-actions exist in several standard examples as discussed in §7.

We thus begin by introducing the notion of a differentiable $\boldsymbol{R}_+$-action G in a multiplicatively convex Fréchet algebra A (see §1). However, most formal constructions below will be carried out without referring to the algebra structure of A. The assumption of A being an algebra is mainly to reflect some important cases. The differentiable $\boldsymbol{R}_+$-action in A leads us to define a scale $\{B^\rho ; \rho\in\boldsymbol{R}\}$ of Fréchet spaces, and the spaces Γ^μ, $\mu\in\boldsymbol{C}$, of G-homogeneous elements (see §2). We then construct the analogues of the spaces of formal series, C^μ, from Γ^μ's. We can thus introduce the notions of developable elements and their developments, as generalizations of elements admitting asymptotic expansions and their expansions. The spaces D^μ of developable elements are shown to be Fréchet spaces. The mappings α^μ, assigning to each element in D^μ its development in C^μ, are then continuous (see §3). Sufficient conditions on surjectivity of α^μ will be discussed in §5. Of course, in such a general situation, α^μ are not necessarily surjective (see Example 7.5). The spaces D^μ are characterized in terms of the boundary behavior of the differentiable $\boldsymbol{R}_+$-action. This permits us to write down the mappings α^μ as a variant of the Taylor expansion (see §4, Theorem 4.5 in particular). We supplement in §6 the cases when A is a Fréchet Montel space.