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The orbits of affine symmetric spaces under the action of minimal parabolic subgroups

By Toshihiko MATSUKI

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Introduction

An affine symmetric space is a triple (G, H, σ) consisting of a connected Lie group G, a closed subgroup H of G and an involutive automorphism σ of G such that H lies between G_{σ} and the identity component of G_{σ}, where G_{σ} denotes the closed subgroup of G consisting of all the elements fixed by σ. Suppose that G is real semi-simple. We are interested in the double coset decomposition $H\backslash G/P$, where P is a minimal parabolic subgroup of G. These double cosets are considered as H-orbits on G/P or as P-orbits on $H\backslash G$.

If H is a maximal compact subgroup of G (when G is of finite center) and σ is the corresponding Cartan involution, this orbit structure is trivial in view of the Iwasawa decomposition $G=KA_{\mathfrak{p}}N^{+}$, where $P=MA_{\mathfrak{p}}N^{+}$ and $H=K$. If the affine symmetric space is $(G\times G, \Delta G, \sigma)$ where G is real semi-simple, ΔG denotes the diagonal of $G\times G$ and σ is the mapping $(x, y)\rightarrow(y, x)$, then the orbit structure can be easily reduced to the Bruhat decomposition $G=\bigcup_{w\in W} PwP$. In the case of (G_{C}, G, σ), where G_{C} is a complex semi-simple Lie group, G is a real form of G_{C} and σ is the conjugation of G_{C} with respect to G, then the orbit structure is studied in Aomoto [1] and Wolf [8].

In this paper the orbit structure is determined for an arbitrary affine symmetric space such that G is real semi-simple.

Let (G, H, σ) be an affine symmetric space such that G is real semi-simple, and $(\mathfrak{g}, \mathfrak{h}, \sigma)$ the corresponding symmetric Lie algebra. Let θ be a Cartan involution commutative with σ (cf. Berger [2]), and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ the corresponding Cartan decomposition. Since the factor space G/P is identified with the set of all the minimal parabolic subalgebras of $\mathfrak{g}$, the following theorem and corollary which are the extension of [1] Theorem 3 and of [8] 2.6 Theorem give a complete characterization of H-orbits on G/P.

THEOREM 1. (i) *Let $\mathfrak{P}$ be a minimal parabolic subalgebra of $\mathfrak{g}$. Then there exists a σ-stable maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ and a positive system Σ^{+} of the root system Σ of the pair $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ such that $\mathfrak{P}$ is H_{0}-conjugate to $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^{+})$ (where H_{0} is the identity component of H, $\mathfrak{P}(\mathfrak{a}_{\mathfrak{p}}, \Sigma^{+})=\mathfrak{m}+\mathfrak{a}_{\mathfrak{p}}+\mathfrak{n}^{+}$, $\mathfrak{m}=\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, $\mathfrak{n}^{+}=\sum_{\alpha\in\Sigma^{+}}\mathfrak{g}_{\alpha}$, and*