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Comparison theorems for Banach spaces of solutions of $\Delta u=Pu$ on Riemann surfaces

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§ 1. Introduction.

Let R be an open Riemann surface and P a density on R, that is, a non-negative Hölder continuous function on R which depends on the local parameter $z=x+iy$ in such a way that the partial differential equation

(1.1) $$\Delta u=Pu\ ,\quad \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2\ ,$$

is invariantly defined on R. A real valued function f is said to be a *P-harmonic* function in an open set U of R, if f has continuous partial derivatives up to the order 2 and satisfies the equation (1.1) on U. The totality of bounded P-harmonic functions on R is denoted by $PB(R)$. Then, $PB(R)$ is a Banach space with the uniform norm

(1.2) $$\|f\|=\sup_{z\in R}|f(z)|\ .$$

H. L. Royden [1] studied the comparison problem of Banach space structures of $PB(R)$ for different choices of densities P on a hyperbolic Riemann surface R and proved the following comparison theorem: If P and Q are non-negative densities on R such that there is a constant $c\geqq 1$ with

(1.3) $$c^{-1}Q\leqq P\leqq cQ$$

outside some compact subset of R, then the Banach spaces $PB(R)$ and $QB(R)$ are isomorphic. On the other hand, concerning this comparison problem M. Nakai [1] gave a different criterion for $PB(R)$ and $QB(R)$ to be isomorphic and proved the following theorem: If two densities P and Q on R satisfy the condition

(1.4) $$\int_R|P(z)-Q(z)|\{G^P(z, w_1)+G^Q(z, w_0)\}\,dxdy<+\infty$$

for some points w_0 and w_1 in R, where $G^P(z, w)$ and $G^Q(z, w)$ are Green's functions of R associated with (1.1) and the equation $\Delta u=Qu$ respectively, then Banach spaces $PB(R)$ and $QB(R)$ are isomorphic.