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A characterization of crossed products of factors by discrete outer automorphism groups

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§ 1. Introduction.

Let $\mathcal{A}$ be a factor, $\mathcal{G}$ a discrete countable group and α a faithful representation of $\mathcal{G}$ into $Out(\mathcal{A})$ (the group of all outer automorphisms of $\mathcal{A}$). Then it is known that there is a faithful normal expectation of the crossed product $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha)$ of $\mathcal{A}$ by $\mathcal{G}$ onto $\mathcal{A}$ and that the relative commutant $\mathcal{A}' \cap \mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha)$ is the scalar multiples of the identity operator (cf, [**3**; Lemma 4]). In [**1**], we considered the crossed product $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ of a general von Neumann algebra $\mathcal{A}$ by a discrete countable group $\mathcal{G}$ with a factor set $\{v(g, h); g, h \in \mathcal{G}\}$ of unitaries in $\mathcal{A}$ and showed that there is a faithful normal expectation of $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ onto $\mathcal{A}$, where α is a semirepresentation of $\mathcal{G}$ into $Aut(\mathcal{A})$ (the group of all automorphisms of $\mathcal{A}$). Using this expectation, we can show that the relative commutant of $\mathcal{A}$ in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ is contained in the center of $\mathcal{A}$ if α_g $(g \neq 1)$ is freely acting on $\mathcal{A}$ (Theorem 2), so that the relative commutant of a factor $\mathcal{A}$ in $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ is the scalar multiples of the identity operator if α is a semirepresentation of $\mathcal{G}$ into $Out(\mathcal{A})$ (Corollary 3). Is the converse of this result true? The purpose of this paper is to show that the converse of this result is true. That is, for a von Neumann algebra $\mathcal{M}$ generated by the normalizer of a subfactor $\mathcal{A}$ if there is a faithful normal expectation of $\mathcal{M}$ onto $\mathcal{A}$ and the relative commutant of $\mathcal{A}$ in $\mathcal{M}$ is the scalar multiples of the identity operator, then there exists a discrete countable group $\mathcal{G}$ such that $\mathcal{M}$ is isomorphic to $\mathcal{R}(\mathcal{G}, \mathcal{A}, \alpha, v)$ for some semirepresentation α of $\mathcal{G}$ into $Out(\mathcal{A})$ (Theorem 4).

§ 2. Crossed products with factor sets.

Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{G}$ a countable discrete group. A mapping α of $\mathcal{G}$ into $Aut(\mathcal{A})$ is called a *semirepresentation* if, for each g and h in $\mathcal{G}$, there exists an inner automorphism $\iota(g, h)$ of $\mathcal{A}$ such that

$$\alpha_g \alpha_h = \alpha_{gh} \iota(g, h) \qquad (g, h \in \mathcal{G}). \tag{1}$$