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On the least positive eigenvalue of the Laplacian for compact group manifolds

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§ 1. Introduction.

Let M be an n-dimensional compact connected manifold. For every Riemannian metric g on M, let $-\Delta_g$ be the Laplace-Beltrami operator acting on smooth functions on M. Let $\lambda_1(g)$ be the least positive eigenvalue of Δ_g. M. Berger ([1] p. 138) posed the problem: Does there exist a positive constant $k(M)$ such that

$$\lambda_1(g)\,\mathrm{vol}\,(M, g)^{2/n}\leqq k(M)\,,$$

for every Riemannian metric g on M? J. Hersch [5] showed that if M is diffeomorphic with the 2-dimensional sphere S^2, then for every Riemannian metric g on M,

$$\lambda_1(g)\,\mathrm{area}\,(S^2, g)\leqq 8\pi\,.$$

The equality holds if and only if (S^2, g) is the canonical sphere.

In the present paper, let M be a compact connected. Lie group. Let us consider the problem: Does there exist a positive constant $k(M)$ such that

$$\lambda_1(g)\,\mathrm{vol}\,(M, g)^{2/n}\leqq k(M)$$

for every *left invariant* Riemannian metric g on M? For this problem we claim (cf. theorem 4) the following: *The only compact Lie group M which has a positive answer for this problem is a torus* $\boldsymbol{T}^n$, that is, *if the compact connected Lie group M has a non-trivial commutator subgroup, then there exists a family of left invariant Riemannian metrics* $g(t)\,(0<t<\infty)$ *on* M *such that* $\lim_{t\to\infty}\lambda_1(g(t))=\infty$, $\lim_{t\to 0}\lambda_1(g(t))=0$ *and* $\mathrm{vol}\,(M, g(t))$ *is constant in* t. In particular, since $SU(2)$ (resp. $SO(3)$) is diffeomorphic with S^3 (resp. $P^3(\boldsymbol{R})$), the above shows that M. Berger's conjecture is negative for S^3 and $P^3(\boldsymbol{R})$. It is known (cf. [1]) that, for a torus $\boldsymbol{T}^n$, there exists a positive constant $k(\boldsymbol{T}^n)$ such that $\lambda_1(g)\,\mathrm{vol}\,(\boldsymbol{T}^n, g)^{n/2}\leqq k(\boldsymbol{T}^n)$ for every left invariant Riemannian metric g on $\boldsymbol{T}^n$.

In § 2, we shall express the Laplace-Beltrami operator on a connected Lie group in term of the left invariant vector fields. In § 3, we shall give an estima-