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Tight spherical designs, I

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§ 1. Introduction.

Let $\boldsymbol{R}^d$ be Euclidean space of dimension d and Ω_d the set of unit vectors in $\boldsymbol{R}^d$. A non-empty finite set $X \subseteqq \Omega_d$ is called a *spherical t-design* in Ω_d if

$$\sum_{\alpha \in X} W(\alpha)=0$$

for all homogeneous harmonic polynomials W on $\boldsymbol{R}^d$ of degree $1, 2, \cdots, t$. This is equivalent to the condition that the k-th moments of X are invariant under orthogonal transformations of $\boldsymbol{R}^d$ for $k=0, 1, 2, \cdots, t$. These designs were studied by Delsarte, Goethals and Seidel [4]. They proved that the cardinality of a design is bounded below;

$$|X| \geqq \binom{d+n-1}{d-1}+\binom{d+n-2}{d-1} \qquad \text{if} \quad t=2n,$$

$$|X| \geqq 2\binom{d+n-1}{d-1} \qquad \text{if} \quad t=2n+1.$$

They called a design *tight* if it attains this bound. They constructed examples of tight spherical t-designs for $t=2, 3, 4, 5, 7, 11$, and proved ([4], Theorem 7.7) that no such designs exist for $t=6$, except the regular heptagon in Ω_2. Bannai [1] proved that for given $t \geqq 8$, there exist tight spherical designs in Ω_d for only finitely many values of d.

In this paper we will prove

THEOREM 1. *Let* $t=2n$ *and* $n \geqq 3$ *and* $d \geqq 3$. *Then there exists no tight spherical t-design in* Ω_d.

In a subsequent paper we hope to prove a similar result when t is odd. Note that if $d=2$ the only tight spherical design is the regular $(t+1)$-gon.

The proof is similar to that of Theorem 7.7 in [4], which is the special case $t=6$. We first prove that if a design exists, then a certain polynomial (written $R_n(x)$, defined in § 2 below) has all its roots rational. By reducing $R_n(x)$ modulo various primes, we show that if its roots are all rational, then