

On Kähler fiber spaces over curves*

By Takao FUJITA

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Introduction.

(0.0) In the study of higher-dimensional varieties we often face such questions: Let $f: M \rightarrow S$ be a fiber space. Suppose that S and a general fiber of f enjoy such and such properties. Then how is M ?

One of the most important problems of this type is the addition conjecture for Kodaira dimensions (See [10], [16]). Recent developments in the classification theory of algebraic varieties (see [17], [18], [2], [17a]) throw light upon the relation between the above conjecture and 'positivity' of $f_*\omega_{M/S}$. In this paper we prove the numerical semi-positivity of $f_*\omega_{M/S}$ in case S is a curve.

(0.1) To be precise we fix our notation and terminology. *Variety* means an irreducible reduced compact complex analytic space. *Manifold* means a smooth variety. *Fiber space* is a triple (f, M, S) , where f is a surjective morphism $M \rightarrow S$ whose general fiber is connected. Moreover, M and S are assumed to be smooth unless otherwise stated explicitly. This fiber space is said to be *Kähler* (resp. *projective*) if so is M . For a locally Macaulay variety V , ω_V denotes the dualizing sheaf of it (for the duality theory, see [7], [14], [15]). For a fiber space $f: M \rightarrow S$ we denote by $\omega_{M/S}$ the relative dualizing sheaf $\omega_M \otimes f^*\omega_S^\vee \cong \mathcal{O}_M(K_M - f^*K_S)$, where K_X denotes the canonical bundle of a manifold X .

The following three facts are well-known.

(0.2) ω_V is torsion free for any locally Macaulay variety V .

(0.3) $g_*\mathcal{F}$ is torsion free for any surjective morphism $g: X \rightarrow Y$ and for any torsion free sheaf \mathcal{F} on X .

(0.4) Any torsion free sheaf on a smooth curve is locally free.

(0.5) Combining the above facts we infer that $f_*\omega_{M/C}$ is locally free for any fiber space $f: M \rightarrow C$ over a curve C . Moreover $\text{rank } f_*\omega_{M/C} = p_g(F) = h^{n,0}(F)$ where F is a general fiber of f and $n = \dim F$.

(0.6) MAIN THEOREM (see (2.7)). $f_*\omega_{M/C}$ is numerically semi-positive for any Kähler fiber space over a curve C . Namely, the invertible sheaf $\mathcal{O}(1)$ on

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