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Zero-divisors of character rings of finite groups

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Introduction.

In [**9**] Roquette gives a decomposition of 1 in $R_\lambda(G)$ into a sum of primitive idempotents, where G is a finite group and $R_\lambda(G)$ denotes the character ring of G with coefficients in $\mathfrak{p}$-adic integers λ. On the other hand, Serre [**10**] has shown that the prime spectrum $\mathrm{Spec}(R(G))$ of the character ring $R(G)$ is connected with respect to the Zariski topology, that is, $R(G)$ has no non-trivial idempotent.

This paper aims at extending these results to the case where the coefficient ring λ is a Dedekind domain in the complex number field. It is shown in the section 3 that if every non-zero prime ideal contains a prime number, then it is necessary and sufficient for $\mathrm{Spec}(R_\lambda(G))$ to be connected that no prime divisor of the order of G is a unit in λ (Corollary 1 to Proposition 6). From this result a characterization of finite p-groups is given in Theorem 3. In particular, G is a p-group if and only if $\mathrm{Spec}(R_\lambda(G))$ is connected when λ is a discrete valuation ring in which p is a non-unit.

The main step in the proofs of these results is to find special zero-divisors of $R_\lambda(G)$ (Theorem 2), and this is done by using the ideas of [**9**] and [**11**]. These zero-divisors are also used to prove a converse of some result due to Atiyah [**1**].

The above results contain the corresponding results for a finite abelian group ring, since in this case the group ring is isomorphic to the character ring. Swan [**13**, Corollary 8.1] has shown that the group ring $\lambda[G]$ has no non-trivial idempotent if λ is a Dedekind domain of characteristic 0 and no prime divisor of the order of G is a unit in λ. If G is abelian, then it follows from Proposition 5 that G is a p-group if $R(G)$ is Hausdorff with respect to the augmentation topology (see §3.1). This is a special case of Sinha [**12**, Corollary].

The section 1 of this paper deals with the prime ideals of $R_\lambda(G)$ for an arbitrary ring λ contained in the complex number field. As an analogue of [**6**, §2, h) and i)], Proposition 1 gives a necessary and sufficient condition for $R_\lambda(G)$ to be a local ring. Moreover some zero-divisors of $R_\lambda(G)$ are constructed