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Homeomorphisms on a three dimensional handle

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McMillan proved that any two sets of generators for $\pi_1(H)$ are equivalent for an orientable handle H. We extend his result to the non-orientable case. These results may be interesting in view of non-orientable Heegaard diagrams of closed 3-manifolds, in particular $P^2 \times S^1$ which has that of genus two. All manifolds considered are to be triangulated. All embeddings and homeomorphisms are to be piecewise linear.

DEFINITION. Let H be a compact connected 3-manifold. We say that H is an orientable or non-orientable handle with genus n respectively when H is homeomorphic to $D_1^2 \times S^1 \sharp \cdots \sharp D_n^2 \times S^1$ or $M_1^2 \times I \sharp \cdots \sharp M_n^2 \times I$ where D_i^2 is a 2-disk, S^1 is a 1-sphere, M_i^2 is a Mobius band, I is a unit interval and $\sharp$ is a disk sum (boundary connected sum).

Note that $D^2 \times S^1 \sharp M^2 \times I$ is homeomorphic to $M^2 \times I \sharp M^2 \times I$.

DEFINITION. Let H be a handle with genus n and $J_1, \cdots, J_n$ mutually disjoint simple closed curves on ∂H. We say that $\{J_k\}_{k=1}^n$ is a system of generators for $\pi_1(H)$ when S is connected and the inclusion homomorphism $\pi_1(S) \to \pi_1(H)$ is onto where $S = \partial H - \bigcup_{k=1}^{n} \mathring{N}(J_k, \partial H)$ and $N(J_k, \partial H)$ is a regular neighborhood of J_k's in ∂H. (Compare the definition in [3].)

DEFINITION. Let $\{J_i\}_{i=1}^n$, $\{\tilde{J}_k\}_{k=1}^n$ be two systems of generators for $\pi_1(H)$. We say that $\{J_i\}_{i=1}^n$ is equivalent to $\{\tilde{J}_k\}_{k=1}^n$ when there is a homeomorphism of H onto H throwing the elements of $\{J_k\}_{k=1}^n$ onto those of $\{\tilde{J}_i\}_{i=1}^n$.

DEFINITION. Let M be a compact 3-manifold. We say that M is irreducible when any 2-sphere embedded in M bounds a 3-cell in M.

Hereafter let M be a compact connected 3-manifold such that ∂M is non-empty.

DEFINITION. Let L be a simple closed curve in M. Then the curve L is said to be orientable (resp. non-orientable) if $N(L, M)$ is homeomorphic to $D^2 \times S^1$ (resp. $M^2 \times I$ where M^2 is a Mobius band).

LEMMA 1. *If M is irreducible and $\pi_1(M)$ is n-free, then ∂M is connected.*

PROOF. The proof is by induction on the rank of $\pi_1(M)$. If $\pi_1(M) = \{0\}$, then each component of ∂M is a 2-sphere and so all the 2-spheres bound 3-cells.