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L^p-spaces and maximal unbounded Hilbert algebras

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§ 0. Introduction.

Inductive and projective limits of the L^p-spaces with respect to a Hilbert algebra are studied. By using their spaces we give necessary and sufficient conditions under which a maximal unbounded Hilbert algebra defined in [11] is pure.

In this paper $\mathcal{D}_0$ denotes a Hilbert algebra, $\mathfrak{h}$ the completion of $\mathcal{D}_0$, $\mathcal{U}_0(\mathcal{D}_0)$ the left von Neumann algebra of $\mathcal{D}_0$, ϕ_0 the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$ and π_0 the left regular representation of $\mathfrak{h}$.

In [11~12], we have studied unbounded Hilbert algebra which is a generalization of the notion of Hilbert algebra to unbounded case. Let $L^p(\phi_0)$ be the L^p-space with respect to ϕ_0 and let $\|T\|_p$ be the L^p-norm of $T\in L^p(\phi_0)$. The space $L_2^\omega(\mathcal{D}_0)$ defined by:

$$L_2^\omega(\mathcal{D}_0)=\bigcap_{2\leq p<\infty} L_2^p(\mathcal{D}_0) \qquad (\text{where} \quad L_2^p(\mathcal{D}_0):=\{x\in\mathfrak{h}\,;\,\overline{\pi_0(x)}\in L^p(\phi_0)\})$$

is maximal among unbounded Hilbert algebras containing $\mathcal{D}_0$ and it plays an important role for our study of unbounded Hilbert algebras.

In this paper we shall investigate the space $L_2^\omega(\mathcal{D}_0)$ by using the L_2^p-spaces and inductive, projective limits of L_2^p-spaces.

Under the norm $\|x\|_{(2,p)}:=\max(\|x\|_2, \|x\|_p)$ (where $\|x\|_p:=\|\overline{\pi_0(x)}\|_p$), $L_2^p(\mathcal{D}_0)$ is a Banach space. Furthermore,

$$\mathfrak{h}\supset L_2^p(\mathcal{D}_0)\supset L_2^q(\mathcal{D}_0)\supset L_2^\omega(\mathcal{D}_0)\supset L_2^\infty(\mathcal{D}_0) \quad (2<p<q<\infty).$$

We define

$$L_2^{p-}(\mathcal{D}_0)=\bigcap_{2\leqq t<p} L_2^t(\mathcal{D}_0) \quad (2<p\leqq\infty),$$

$$L_2^{p+}(\mathcal{D}_0)=\bigcup_{t>p} L_2^t(\mathcal{D}_0) \quad (2\leqq p<\infty)$$

and give $L_2^{p-}(\mathcal{D}_0)$ (resp. $L_2^{p+}(\mathcal{D}_0)$) the projective limit topology τ_2^{p-} (resp. the inductive limit topology τ_2^{p+}) for the Banach spaces $(L_2^t(\mathcal{D}_0)\,;\,\|\ \|_{(2,t)})$. Then it is proved that $(L_2^{p-}(\mathcal{D}_0)\,;\,\tau_2^{p-})$ is a Fréchet space, $L_2^{\infty-}(\mathcal{D}_0)=L_2^\omega(\mathcal{D}_0)$ and $(L_2^{p+}(\mathcal{D}_0)\,;\,\tau_2^{p+})$ is a separated barrelled space.