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Product theorem of the fundamental group of a reducible curve

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1. In this note, we study the fundamental groups of the complement of reducible curves and we prove the following generalization of the result of [2].

THEOREM. *Let C_1 and C_2 be plane algebraic curves in $\boldsymbol{C}^2$. Assume that the intersection $C_1 \cap C_2$ consists of distinct $d_1 d_2$ points where d_i $(i=1, 2)$ are respective degrees of C_1 and C_2. Then the fundamental group $\pi_1(\boldsymbol{C}^2 - C_1 \cup C_2)$ is isomorphic to the product of $\pi_1(\boldsymbol{C}^2 - C_1)$ and $\pi_1(\boldsymbol{C}^2 - C_2)$.*

2. PROOF.

Let (x, y) be a coordinate of $\boldsymbol{C}^2$, and let $f(x, y)$ and $g(x, y)$ be defining polynomials of C_1 and C_2 respectively. We can assume that the x-axis and y-axis are in general position with respect to C_1 and C_2. Consider the deformations $C_1(t)$ and $C_2(\tau)$ $(t, \tau \in \boldsymbol{C})$ of C_1 and C_2 defined by,

$$C_1(t):\ f(x, ty)=0\,,$$

$$C_2(\tau):\ g(\tau x, y)=0\,.$$

Obviously, each deformation is biholomorphic if $t \neq 0$ or $\tau \neq 0$, and $C_i(1)=C_i$ $(i=1, 2)$, so that $\boldsymbol{C}^2 - C_i(t)$ is homeomorphic to $\boldsymbol{C}^2 - C_i$ for all $t \neq 0$. The intersection $C_1(t) \cap C_2(\tau)$ consists of distinct $d_1 d_2$ points for $(t, \tau) \in U$ where U is a Zariski open set of $\boldsymbol{C}^2$. For any (t_0, τ_0) in U we can construct a one parameter family of curves $\{C_1(t(s)) \cup C_2(\tau(s))\,;\ 0 \leqq s \leqq 1\}$ such that $(t(s), \tau(s))$ is contained in U for each $0 \leqq s \leqq 1$, and $t(0)=\tau(0)=1$, $t(1)=t_0$, $\tau(1)=\tau_0$. Hence, $\boldsymbol{C}^2 - C_1 \cup C_2$ is homeomorphic to $\boldsymbol{C}^2 - C_1(t_0) \cup C_2(\tau_0)$. (See [2], for the precise proof.) So it is enough to show that $\pi_1(\boldsymbol{C}^2 - C_1(t_0) \cup C_2(\tau_0))$ is isomorphic to the product of $\pi_1(\boldsymbol{C}^2 - C_1(t_0))$ and $\pi_1(\boldsymbol{C}^2 - C_2(\tau_0))$, for a suitable $(t_0, \tau_0) \in U$.

The curve $C_1(0)$ consists of distinct d_1 lines which are parallel to the y-axis, and $C_2(0)$ consits of distinct d_2 lines which are parallel to the x-axis, because, by the assumption, the equations $f(x, 0)=0$ and $g(0, y)=0$ have distinct d_1 and d_2 roots respectively. We consider the following parallel lines: $L_\lambda:\ y=x+\lambda$ $(\lambda \in \boldsymbol{C})$. For a fixed general λ_0, we can take loops a_j $(j=1, \cdots, d_1)$ and b_k $(k=1, \cdots, d_2)$ generating $\pi_1(L_{\lambda_0} - L_{\lambda_0} \cap (C_1(0) \cup C_2(0)))$, so that $[a_j, b_k]=a_j b_k a_j^{-1} b_k^{-1}$ becomes the unit element in $\pi_1(\boldsymbol{C}^2 - C_1(0) \cup C_2(0))$. Here a_j (respectively b_k) is a small