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Volume estimate of submanifolds in compact Riemannian manifolds

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0. Introduction.

In a study of a given Riemannian manifold $\bar{M}$, it is important and also interesting by itself to know a precise value of the injectivity radius of $\bar{M}$. Here the injectivity radius $i(\bar{M})$ of $\bar{M}$ is, by definition, the supremum of a number t such that every geodesic in $\bar{M}$ with length$<t$ is the shortest connection between its end points. And in general $i(\bar{M})$ can be estimated from below by using a number which relates to half of the infimum of length of all closed geodesics in $\bar{M}$ and hence it is needed to know the infimum of length of all closed geodesics in $\bar{M}$ for the estimate of $i(\bar{M})$. As a remarkable result in this field, J. Cheeger in [2] gave a lower bound of length of all closed geodesics in $\bar{M}$ depending on the volume, the sectional curvature and the diameter of $\bar{M}$.

Now if we consider a closed geodesic in $\bar{M}$ as a 1-dimensional compact totally geodesic submanifold of $\bar{M}$, then the problem to estimate length of closed geodesics in $\bar{M}$ can be generalized as follows. "Is it possible to estimate the volume of compact totally geodesic submanifolds of $\bar{M}$ by using the geometrical terms of $\bar{M}$?". Then from this point of view, a result obtained by N. Grossman in [5], which gives an estimate of the volume of totally geodesic hypersurface in a certain pinched manifold, may be regard as a partial answer to this problem. And with respect to the Grossman's result, we can give a slight generalization, see Theorem A. In Section 3, we will give an answer to the problem mensioned above in a more generalized form. Namely the volume of a compact submanifold M of $\bar{M}$ is estimated from below by using the principal curvatures of the second fundamental forms on M. Furthermore when the codimension of M is 1, a lower bound for the volume of M is given in a further generalized form in the sense that the mean curvature, the mean value of the principal curvatures, is used instead of the principal curvatures. This is shown in Section 2. We will give an upper bound for the volume of M with codimension one or two in Sections 2 and 4.