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Some differential equations on Riemannian manifolds

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§ 1. Introduction.

Let (M, g) be a Riemannian manifold of dimension $m \geqq 2$ and let ∇ denote the Riemannian connection defined by g. In this paper we study the following system of differential equations of order three:

$$\nabla_h \nabla_j \nabla_i f + k(2\nabla_h f g_{ji} + \nabla_j f g_{ih} + \nabla_i f g_{hj}) = 0 \tag{1.1}$$

where k is a positive constant. Originally the differential equations (1.1) come from some study of the Laplacian on a Euclidean sphere $(S^m; k)$ of constant curvature k. The first eigenvalue of the Laplacian on $(S^m; k)$ is mk and each eigenfunction h corresponding to mk satisfies the following system of differential equations of order two:

$$\nabla_j \nabla_i h + khg_{ji} = 0. \tag{1.2}$$

The second eigenvalue is $2(m+1)k$ and each eigenfunction f corresponding to $2(m+1)k$ satisfies (1.1).

Assuming the existence of a non-constant function h satisfying (1.2) on a Riemannian manifold (M, g) many mathematicians studied differential geometric properties of (M, g) (cf. S. Ishihara and Y. Tashiro [**11**], M. Obata [**14**], [**15**], Y. Tashiro [**22**], etc.). In this case grad f is an infinitesimal conformal transformation.

Assume that there is a non-constant function f satisfying (1.1) on (M, g). Then grad f is an infinitesimal projective transformation and is a k-nullity vector field on (M, g). The converse is also true (cf. Proposition 2.1). This gives a geometric meaning of (1.1).

The system of differential equations (1.1) was first studied by M. Obata [**15**] and he announced the following.

THEOREM A. *Let (M, g) be a complete and simply connected Riemannian manifold. In order for (M, g) to admit a non-constant function f satisfying* (1.1)

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