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On the hypoellipticity for infinitely degenerate semi-elliptic operators

By Yoshinori MORIMOTO

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§ 0. Introduction.

In this paper we shall study hypoellipticity for a partial differential operator of the form

$$P=a(x, y, D_x)+g(x)b(x, y, D_y) \quad \text{in} \quad R^n=R_x^{n_1}\times R_y^{n_2}, \tag{0.1}$$

where $a(x, y, D_x)$ and $b(x, y, D_y)$ are strongly elliptic operators of order $2l$ and $2m$ with respect to x and y, respectively, and $g(x)$ is a smooth non-negative function with a zero point of infinite order at $x=0$ in $R_x^{n_1}$. The operator of the form $(-\Delta_x)^l+g(x)(-\Delta_y)^m$ is a typical example.

Our main theorem is roughly stated as follows: Assume that $b(x, y, D_y)$ is of second order (, but $a(x, y, D_x)$ is not necessarily of second order). Then we have the statement:

$$(*) \qquad \text{“}u\in \mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_s^{loc}(\Omega)\text{”} \qquad \text{for any } \Omega\subset R^n,$$

and therefore P is hypoelliptic in R^n. When $b(x, y, D_y)$ is of higher order $\geqq 4$, we set the following condition on $g(x)$.

Condition (G).

$$|\partial_x^\beta g(x)|\leqq C_\beta g(x)^{1-\sigma|\beta|} \qquad \text{in a neighborhood of} \quad x=0$$

for a fixed σ $(0<\sigma<\{2(m+l(m-1))\}^{-1})$. Then, we have the statement (*) in this case, too. (Such a σ is determined from Propositions 5.1 and 5.2. See Remark of Proposition 5.2.)

When $g(x)$ has a zero point of finite order, fairly complete results have been obtained by Hörmander [8], [9], Grushin [6], [7], Beals [1], Y. Kato [11], Kumano-go-Taniguchi [15], Taniguchi [17], Tsutsumi [19], etc. In such case except [17] we have the stronger result than (*), that is, the statement

$$(**) \qquad \text{“}u\in \mathcal{D}'(\Omega),\quad Pu\in H_s^{loc}(\Omega)\Rightarrow u\in H_{s+\sigma_0}^{loc}(\Omega)\text{”} \qquad \text{for any} \quad \Omega\subset R^n$$

holds for some positive number σ_0. It should be noted that we can no longer expect the statement (**) for the operator of the form (0.1) when $g(x)$ has a zero point of infinite order (see Theorem 1.2).