J. Math. Soc. Japan
Vol. 30, No. 2, 1978

On the values of Hecke's L-functions at non-positive integers

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(Received April 9, 1976)
(Revised July 8, 1977)

§ 0. Introduction.

Let F be a totally real algebraic number field of finite degree g, and $\mathfrak{m}$ be an integral ideal of F. Let χ be a character of finite order of the narrow ray class group modulo $\mathfrak{m}$. Then we say that χ is totally real (resp. totally imaginary) if the field corresponding to the kernel of χ is totally real (resp. totally imaginary). Consider the L-function with character χ which is defined as usual by $L(s, \chi)=\sum_{(\mathfrak{a},\mathfrak{m})=1}\chi(\mathfrak{a})(N(\mathfrak{a}))^{-s}$, where the summation runs over all integral ideals $\mathfrak{a}$ of F prime to $\mathfrak{m}$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$ from F to $\boldsymbol{Q}$. In his paper [**12**], C. L. Siegel considered the ζ-function $\zeta(\mathfrak{b}, \mathfrak{m}, s)=\sum_{\mathfrak{g}}(N(\mathfrak{g}))^{-s}$, where the summation runs over all integral ideals $\mathfrak{g}$ in the same narrow ray class mod. $\mathfrak{m}$ as a fixed ideal $\mathfrak{b}$ prime to $\mathfrak{m}$, and showed that there is an algorithm to compute $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ for positive integers m, and that the value $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ is contained in the rational number field $\boldsymbol{Q}$. Especially, when $\mathfrak{m}=\mathfrak{O}_F$ (the maximal order of F), he obtained an arithmetical expression of $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ (see Siegel [**11**]). Recently, using a method essentially different from Siegel's, T. Shintani [**10**] has given a formula for the value $\zeta(\mathfrak{b}, \mathfrak{m}, 1-m)$ (hence that of $L(1-m, \chi)$ without our assumptions on χ (see below)) in terms of (a generalization of) Bernoulli polynomials in several variables.

In this paper, we shall introduce a certain trick which enables us to apply Siegel's Theorem (see Siegel [**11**] Satz 1, and see text Lemma 2.1) in the case of a non-trivial character χ which satisfies the following conditions:

(i) *χ is totally real or totally imaginary,*

(ii) *$\mathfrak{m}\neq 1$, and $\mathfrak{m}\cap\boldsymbol{Z}$ is a prime ideal $p\boldsymbol{Z}$.*

When the restriction of χ to $\boldsymbol{Z}$ is not trivial, we shall further assume a certain additional not too restrictive condition on χ (for details, see Theorem 2.1 and 2.2). Under the above assumptions on χ, if $k>1$, we shall derive a new formula for $L(1-k, \chi)$ (Theorem 2.2) which is similar to the Siegel formula for $\zeta(1-k)$. In particular, we can prove that

(*) *$L(1-k, \chi)$ is contained in $\boldsymbol{Q}(\chi)$ $(=\boldsymbol{Q}(\chi(a)\,;\, a\in\mathfrak{O}_F))$.*