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On graded rings, I

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Introduction.

In this paper, we study a Noetherian graded ring R and the category of graded R-modules. We consider injective objects of this category and we define the graded Cousin complex of a graded R-module M. These concepts are essential in this paper (see, (1.2.1), (1.2.4) and (1.3.3)).

We say that R is a graded ring defined over a field k, if R is positively graded, $R_0=k$ and if R is finitely generated over k. We denote by $\mathfrak{m}$ the unique graded maximal ideal of R. $\mathfrak{m}=R_+=\bigoplus_{n>0} R_n$. In the latter part of this paper, we treat graded rings defined over k. If R is a graded ring defined over k, the category of graded R-modules has very simple dualizing functor and dualizing module. The dualizing functor is given by $\underline{\mathrm{Hom}}_k(\ \ ,\underline{k})$ and the dualizing module is $\underline{\mathrm{Hom}}_k(R,\underline{k})$ (see, (1.2.7) and (1.2.10)).

Also, in this category, the dual of a graded Noetherian (resp. Artinian) R-module is a graded Artinian (resp. Noetherian) R-module. We need not consider the completion of R.

Let R be a graded ring defined over k and let M be a finitely generated graded R-module. We know that several properties of M are determined by its local cohomology groups $\underline{H}^i_{\mathfrak{m}}(M)$ $(i=0, 1, \cdots)$. For example, M is a Macaulay R-module if and only if $\underline{H}^i_{\mathfrak{m}}(M)=0$ for $i<d=\dim M$ and M is a Gorenstein R-module if and only if $\underline{H}^i_{\mathfrak{m}}(M)=0$ for $i<d$ and $\underline{H}^d_{\mathfrak{m}}(M)$ is an injective R-module. So we study several techniques to calculate local cohomology groups for some operations in the category of graded R-modules (see, (2.2.5), (3.1.1) and (4.1.5)).

The theory of the canonical module of a Noetherian local ring was developed in [15]. We define the canonical module K_R of a graded ring R defined over k as a graded R-module. $K_R=(\underline{H}^d_{\mathfrak{m}}(R))^*$ ($d=\dim R$ and $(\ )^*$ denotes the dual). If R is a Macaulay ring, R is a Gorenstein ring if and only if $K_R\cong R(a)$ for some integer a. This integer $a=a(R)$ is an important invariant of R and plays an essential role in Chapter 3 and Chapter 4 (see, (3.1.5), (3.2.1) and (4.4.7)).

A graded ring R has a geometric object attached to it——Proj(R). If R_+ is generated by R_1, the relationship of R and Proj(R) is treated in [8]. But