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On certain ray class invariants of real quadratic fields

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Introduction.

0-1. In his papers [**10**], [**11**] and [**12**], H. M. Stark introduced certain ray class invariants for totally real fields in terms of the value at $s=0$ of the derivative of some L-series of the fields. Then he presented (with numerical evidences) a striking conjecture on the arithmetic nature of the invariants. In this paper, we show that, for each given real quadratic field, the invariants are described in terms of special values of a certain special function. The function is closely related to the *double gamma function* of E. W. Barnes. Then we prove the conjecture for a very special (but non-trivial) case.

0-2. For a pair $\omega=(\omega_1, \omega_2)$ of *positive* numbers we denote by $\Gamma_2(z, \omega)$ the double gamma function introduced by E. W. Barnes (for the definition and basic properties of the double gamma function, see [**2**] and [**7**]). Set

$$\boldsymbol{F}(z, \omega)=\Gamma_2(z, \omega)/\Gamma_2(\omega_1+\omega_2-z, \omega).$$

Then $\boldsymbol{F}$ is a meromorphic function of z which satisfies the following equalities (0-1) and (0-2).

$$\text{(0-1)} \qquad \boldsymbol{F}(z+\omega_1, \omega)=2\sin(\pi z/\omega_2)\boldsymbol{F}(z, \omega),$$
$$\boldsymbol{F}(z+\omega_2, \omega)=2\sin(\pi z/\omega_1)\boldsymbol{F}(z, \omega).$$

$$\text{(0-2)} \qquad \boldsymbol{F}((\omega_1+\omega_2)/2, \omega)=1.$$

If ω_2/ω_1 is irrational, properties (0-1) and (0-2) characterize $\boldsymbol{F}$ as a meromorphic function of z. Let F be a real quadratic field embedded in the real number field $\boldsymbol{R}$. For an integral ideal $\mathfrak{f}$ of F, denote by $H_F(\mathfrak{f})$ the group of *narrow* ray classes modulo $\mathfrak{f}$ of F. Assume that $\mathfrak{f}$ satisfies the following condition (0-3):

$$\text{(0-3)} \qquad \text{For any totally positive unit } u \text{ of } F,\ u+1\notin\mathfrak{f}.$$

Take a totally positive integer ν of F with the property $\nu+1\in\mathfrak{f}$. Denote by the same letter ν the narrow ray class modulo $\mathfrak{f}$ represented by the principal ideal (ν). Then ν is an element of order 2 of the group $H_F(\mathfrak{f})$. Choose integral ideals $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{h_0}$ of F so that they form a complete set of *narrow* ideal