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Variational inequalities and complementarity problems

By Shigeru ITOH, Wataru TAKAHASHI and Kenjiro YANAGI

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1. Introduction.

We shall consider variational inequalities for multivalued mappings to unify mathematical programming problems and extended fixed point problems. Let X and Y be two real separated topological vector spaces with a given bilinear form $\langle\cdot,\cdot\rangle$ of $Y\times X$ into the reals R. Let T be a multivalued mapping from its domain $D(T)\subset X$ to subsets of Y, f a function from X to R. Under these conditions a solution of a variational inequality is the following; $x_0\in D(T)$ and $w_0\in T(x_0)$ such that $\langle w_0, x-x_0\rangle\geqq f(x_0)-f(x)$ for all $x\in D(T)$. When $D(T)$ is a cone, variational inequalities are related to complementarity problems. Variational inequalities in infinite dimensional spaces were studied by Browder [1], Karamardian [5] and others. Karamardian [5] also considered complementarity problems, for which we also refer to Moré [6]. In this paper we shall give two existence theorems. Using them, we shall solve variational inequalities for multivalued mappings on closed convex subsets in topological vector spaces. Then the results are applied to complementarity problems.

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2. Basic existence theorems.

Let X and Y be two topological spaces. Denote by 2^Y the family of all subsets of Y. A mapping $T: X\to 2^Y$ is said to be upper semicontinuous if $T^{-1}(F)=\{x\in X: T(x)\cap F\neq\emptyset\}$ is closed in X for any closed subset F of Y. The following result was given by Fan [3]. We shall present an elementary proof using Brouwer's fixed point theorem. In the rest of this paper let X and Y be topological vector spaces.

THEOREM 2.1. *Let K be a nonempty compact convex subset of X. Let A be a subset of $K\times K$ for which the following conditions hold:*

(i) *For each $y\in K$, the set $\{x\in K: (x, y)\in A\}$ is closed.*