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The curvatures of the analytic capacity

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§ 1. Introduction.

In [4] Suita has shown that the analytic capacity $c(z)$ of a plane region $D\notin 0_{AB}$ is real analytic and that the curvature of the metric $c(z)|dz|$ is $\leqq -4$. He also raised the conjecture that the curvature is equal to -4 at one point $z\in D$ if and only if $D\in \mathcal{D}_{B}$. D is said to belong to $\mathcal{D}_{B}$ if it is conformally equivalent to the unit disc less (possibly) a closed set expressed as a countable union of compact N_{B} sets. The papers [5] and [2] provide a different proof for Suita's result and actually resolve the conjecture of Suita in case $D\in \mathcal{D}_{p}$, $1\leqq p<\infty$. Here $\mathcal{D}_{p}$ denotes the class of all p-connected regions with no degenerate boundary component. In the present paper we generalize the results of [2] and [5] to higher order curvatures (Theorem 1). Specifically we show that, for any point z in $D\notin 0_{AB}$, $c^{(n+1)^{2}}\leqq(\prod_{k=1}^{n}k!)^{-2}\det\|c_{j\bar{k}}\|_{j,k=0}^{n}$, where $c=c(z)$ and $c_{j\bar{k}}=\dfrac{\partial^{j+k}c}{\partial z^{j}\partial\bar{z}^{k}}$. For $n=1$, we obtain the result of [4]. Moreover, if $D\in \mathcal{D}_{B}$ then we have equality in the above inequality for each $z\in D$ and every $n=0, 1, \cdots$. If $D\in \mathcal{D}_{p}$ then equality at one point $z\in D$ holds if and only if $p=1$. Several other properties related to the analytic capacity are proved. Our proofs are based on the "method of minimum integral" with respect to the Szegö kernel function. As in [2] we also show that the above inequality is strict if the Ahlfors function with respect to z has a zero in D other than z.

§ 2. Analytic capacity.

Let D be a plane region $\notin 0_{AB}$ and let $H(D:\Delta)$ designate the class of all analytic functions from D into the unit disc Δ. Let $\zeta\in D$ and set $H_{\zeta}(D:\Delta)=\{f\in H(D:\Delta):f(\zeta)=0\}$. The analytic capacity $c(\zeta)=c_{D}(\zeta)$ is given by $c(\zeta)=\sup\{|f'(\zeta)|:f\in H_{\zeta}(D:\Delta)\}$. There exists (cf. [3]) a unique function F in $H_{\zeta}(D:\Delta)$, called the Ahlfors function $F(z)=F(z:\zeta)$, such that $F'(\zeta)=c(\zeta)$. Clearly, $c(z)|dz|$ is a conformal invariant metric. Using a canonical exhaustion process (cf. [4]) it can be shown that $c(z)$ is real analytic and hence we can introduce