J. Math. Soc. Japan
Vol. 29, No. 4, 1977

A second order theory of ordinal numbers with Ackermann-type reflection schema

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(Received Aug. 20, 1976)

§ 1. Introduction.

The underlying logic of the ordinal number theory OA given in [3] is a weakened second order logic. Adopting the standard second order logic, we can obtain a stronger theory. We shall denote it by OA^+. In this paper we first show the consistency of OA^+ by interpreting it in ZF. In fact, OA^+ is interpretable in various theories which are much weaker than ZF. Roughly speaking, OA^+ is interpretable in those theories that have the first uncountable ordinal ω_1 and all subsets of $\omega_1 \times \omega_1$. I do not know whether OA^+ is strictly weaker than those theories. Next, we give a theory which is somewhat simple and whose strength is equal to that of OA^+.

§ 2. The theory OA^+.

2.1. The language of OA^+ (denoted by L_0).

(a) Individual variables: $x_0, x_1, \cdots$.

(b) Predicate variables: $P_0, P_1, \cdots$.

(c) Predicate constants: $*=*, *<*, O*$.

(d) Logical symbols: $\neg, \wedge, \exists$.

2.2. The axioms and the inferences of OA^+.

(a) The axioms and the inferences of the standard second order logic and the equality axiom: $a=b \leftrightarrow \forall P[Pa \rightarrow Pb]$.

(b) The following four:

$$Oa \wedge \forall x[x<a \leftrightarrow x<b] \rightarrow a=b\,;$$

$$Oa \wedge x<a \wedge y<x \rightarrow y<a\,;$$

$$\forall P[\forall x[Ox \rightarrow [(\forall y<x)Py \rightarrow Px]] \rightarrow \forall x[Ox \rightarrow Px]]\,;$$

$$Oa_1 \wedge \cdots \wedge Oa_n \wedge \forall x[A(x) \rightarrow Ox \wedge (\forall y<x)A(y)] \rightarrow \exists y[Oy \wedge \forall z[z<y \leftrightarrow A(z)]]\,,$$

where $A(x)$ contains neither the predicate constant O nor free variables except $a_1 \cdots a_n, x$.