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Scattering theory for Schrödinger operators with long-range potentials, I, abstract theory

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§ 1. Introduction.

In the recent development of the scattering theory, the Schrödinger operator $-\Delta+V$ in $\mathfrak{H}=L^2(R^N)$ with a real long-range potential V has been one of the main concerns. Roughly speaking, V is said to be long-range when $V(x)=O(|x|^{-\varepsilon_0})$, $|x|\to\infty$, $\varepsilon_0>0$, while short-range when ε_0 can be taken as $\varepsilon_0>1$. The study of the long-range scattering was initiated by the work of Dollard [7] for the pure Coulomb potential $V(x)=\text{const.}/|x|$. Among other things he showed in [7] that the ordinary wave operator $W^{\pm}=\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2}e^{-itH_1}$, where $H_1=-\Delta$, $H_2=-\Delta+V$, does not exist, but alternatively the modified wave operator $W_D^{\pm}=\underset{t\to\pm\infty}{\text{s-lim}}\, e^{itH_2}e^{-itH_1-iX^{\pm}(t)}$, where $X^{\pm}(\pm t)=\mathcal{F}^{-1}\Big[\int_{\pm 1}^{\pm t}V(s\xi)ds\cdot\Big]\mathcal{F}$ for $t>1$, exists and is complete. Here $\mathcal{F}$ denotes the Fourier transformation in $L^2(R^N)$ and by "complete" it is meant that the range of $W_D^{\pm}$ is equal to the absolutely continuous subspace $\mathfrak{H}_{2,ac}$ of H_2. Dollard's proof of the completeness is based on the eigenfunction expansions of H_2.

After Dollard's work many authors have investigated general long-range potentials and proved the existence of modified wave operators (cf. e. g. Buslaev and Matveev [5], Alsholm and Kato [3], Alsholm [2], Hörmander [8]). It may be said that the study of the existence of modified wave operators has now reached a rather satisfactory stage.

As to the completeness of modified wave operators, however, it seems that no results have yet been published except for spherically symmetric long-range potentials (cf. Amrein, Martin and Misra [4])[1]. In the case of short-range

1) Professor T. Ikebe also obtained a proof of the completeness of time dependent modified wave operators for the case $\varepsilon_0>1/2$, but his method is different from ours mentioned below (a lecture in 1975). Professor S. Agmon also informed the author that he had obtained an eigenfunction expansion theorem and proved the completeness of modified wave operators. His method covers general elliptic operators with general long-range perturbations (private communication). The writer expresses his thanks to Professor Agmon for kindly communicating these results.