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How to differentiate the projection on a convex set in Hilbert space. Some applications to variational inequalities

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Introduction and notations.

We make a local study of the projection onto convex sets in real Hilbert space. Let H be a real Hilbert space, $K\subset H$ a closed convex subset, the projection operator onto K will be denoted by P. For every $u\in K$, we set $S_K(u) = \bigcup_{\lambda>0} \lambda(K-u)$, $\Pi_K(u)=\overline{S_K(u)}$. If $f\in H$, $[f]$=vector space generated by f. If K is a cone with vertex 0, then $K^{\perp}=\{v\in H, \forall f\in K, \langle f, v\rangle\leqq 0\}$. In particular, for $f\in H$, $[f]^{\perp}=\{v\in H, \langle f, v\rangle=0\}$. For K a cone with vertex 0, and $u\in K$, we have

$$S_K(u)=K+[u],\ \Pi_K(u)=\overline{K+[u]}\,.$$

Finally, for K an arbitrary closed convex set, and $v\in H$, we define

$$\Sigma_K(v)=\Pi_K(Pv)\cap[v-Pv]^{\perp}\,.$$

In § 1, we prove under reasonable hypotheses a theorem which shows the role played by the "curvature of the boundary" of K near Pv, for the conical differentiability of P at v. After giving some zoology from geometry or integration, we restrict our attention to the case where $\forall v\in H$, $S_K(Pv)\cap[v-Pv]^{\perp}$ is dense in $\Sigma_K(Pv)$. A convex set that satisfies this property will be called polyhedric. We get the following

THEOREM. *If K is polyhedric, $\forall v\in H$, $\forall z\in H$, then the curve $t\to P(v+tz)$ is strongly right-differentiable at* 0, *with a derivative* $\gamma=\mathrm{Proj}_{\Sigma_K(v)}(z)$.

In § 2, we assume that H is a lattice, with respect to a closed positive cone K. Then K is a polyhedric set under the simple hypothesis that $x\to x^+=\sup\{x, 0\}$ is a bounded map. If $f: [0, T[\to H$ is right-differentiable, then by setting $u(t)=\mathrm{Proj}_K(f(t))$, the preceding theorem gives

$$\forall t\in[0, T[,\ \frac{d^+u}{dt}=\mathrm{Proj}_{\Sigma\ (f(t))}\Big(\frac{d^+f}{dt}\Big)\,.$$