J. Math. Soc. Japan
Vol. 29, No. 3, 1977

Riemannian submersions and critical Riemannian metrics

By Yosio MUTŌ

(Received May 25, 1976)
(Revised Nov. 13, 1976)

Introduction.

In the present paper we consider only Riemannian submersions $\pi:(\tilde{M},\tilde{g})\rightarrow(B,{}^{B}g)$ such that fibers F are complete and connected and imbedded in $(\tilde{M},\tilde{g})$ regularly as totally geodesic submanifolds.

There are many examples of Riemannian manifolds $(\tilde{M},\tilde{g})$ any of which admits such a submersion and at the same time $\tilde{g}$ is a critical Riemannian metric on $\tilde{M}$. But it is in general not true that ${}^{B}g$ is a critical Riemannian metric on B although $\tilde{g}$ is a critical Riemannian metric on $\tilde{M}$. Nevertheless there exist some cases in which $\tilde{g}$ and ${}^{B}g$ are critical Riemannian metrics on $\tilde{M}$ and B simultaneously. A Sasakian manifold $(\tilde{M},\tilde{g},\tilde{\xi})$ admits a Riemannian submersion (Sasakian submersion) $\pi:(\tilde{M},\tilde{g},\tilde{\xi})\rightarrow(B,{}^{B}g)$ such that, if ${}^{B}g$ is an Einstein metric satisfying a subsidiary condition, $\tilde{g}$ and ${}^{B}g$ are simultaneously critical Riemannian metrics or not. If N is a certain integer an N-dimensional sphere (S^{N},g_{0}) with the standard Riemannian metric g_{0} admits one or more Riemannian submersions $\pi:(S^{N},g_{0})\rightarrow(B,{}^{B}g)$ [**2**] and ${}^{B}g$ is always a critical Riemannian metric on B.

On any C^{∞} complete manifold M there can exist various Riemannian metrics g. Among them a critical Riemannian metric is defined as follows if M is compact orientable. Let $\mathcal{M}(M)$ be the space of C^{∞} Riemannian metrics g on M satisfying

$$\int_{M}dV_{g}=1$$

where dV_{g} is the volume element measured by g. Consider a point $g\in\mathcal{M}(M)$. Let $f(K)$ be a scalar field on M determined by g as the contraction of a tensor product of the curvature tensor. Then

$$F_{M}[g]=\int_{M}f(K)dV_{g}$$

defines a mapping $F:\mathcal{M}(M)\rightarrow\boldsymbol{R}$. A critical point of F is denoted by g_{F} and is called a critical Riemannian metric with respect to the field $f(K)$ or the