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Groups of algebras over $A \otimes \bar{A}$

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Introduction.

Let A be an R-algebra, where R is a fixed commutative ring. An *algebra over* A is a pair (U, i) where U is an R-algebra and $i: A \rightarrow U$ an R-algebra map. They form a category. The definition of morphisms is obvious.

Sweedler [1] starts to try to classify algebras over A by their *underlying* A-*bimodules.* In almost all the chapters he assumes the algebra A is *commutative.* His method is useful for such algebras (U, i) over A as i sends A isomorphically onto the *centralizer* of A in U.

When A is commutative, he defines a product "$\times_A$" on the category of algebras over A. This product is neither in general associative nor unitary.

A $\times_A$-*bialgebra* is a triple $(B, \Delta, \mathcal{I})$ where B is an algebra over A and $\Delta: B \rightarrow B \times_A B$, $\mathcal{I}: B \rightarrow \mathrm{End}_R A$ are maps of algebras over A making some diagrams commute.

When Δ is an isomorphism and $\mathcal{I}$ is injective, he defines $\mathcal{E}_B$ to be the set of isomorphism classes of algebras (U, i) over A such that $U \cong B$ as A-*bimodules.* He shows that i then maps A isomorphically onto the centralizer in U of A. The product "$\times_A$" makes $\mathcal{E}_B$ into an abelian monoid with unit $\langle B \rangle$ the class of B.

Let $\mathcal{G}\langle B \rangle$ denote the group of invertible elements in $\mathcal{E}_B$.

Among other things he proves that if $\langle U \rangle$ the class of of U belongs to $\mathcal{G}\langle B \rangle$ then there is a canonical isomorphism of algebras over A

$$\zeta: (U^0 \times_A U)^0 \longrightarrow B$$

with the assumption of the existence of some isomorphism $\mathcal{S}: B \rightarrow (B^0 \times_A B)^0$ of algebras over A, called an "Ess" map. Here we denote by U^0 the *opposite* algebra to U considered as an algebra over A.

Based on this fact, he shows that if A is a simple B-module (via $\mathcal{I}: B \rightarrow \mathrm{End}_R A$), then all algebras (U, i) over A with $\langle U \rangle \in \mathcal{G}\langle B \rangle$ are *simple.* (Exactly, some additional hypothesis on B is needed).

Further, for a $\times_A$-bialgebra $(B, \Delta, \mathcal{I})$ where Δ is an isomorphism and $\mathcal{I}$ is injective he constructs some *semi-co-simplicial complex* consisting of *commutative*