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The degeneracy of systems and the exceptional linear combinations of entire functions

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§ 1. Introduction and preliminaries.

Let $f=(f_0, \cdots, f_n)$ $(n \geqq 1)$ be a transcendental system in $|z|<\infty$. That is, $f_0, \cdots, f_n$ are entire functions without common zero and

$$\lim_{r\to\infty} \frac{T(r, f)}{\log r} = \infty ,$$

where $T(r, f)$ is the characteristic function of f defined by Cartan ([1]).

Let $X=\{F_i ; F_i=\sum_{j=0}^{n} a_{ij} f_j \not\equiv 0\}_{i=0}^{N}$ $(n \leqq N \leqq \infty)$ where a_{ij} are constants and matrices $(a_{i_\nu j})_{j=0,\cdots,n}^{\nu=0,\cdots,n}$ are regular for all $n+1$ integers $\{i_\nu\}_{\nu=0}^{n}$ $(0 \leqq i_\nu \leqq N)$ and λ be the maximum number of linearly independent linear relations among $f_0, \cdots, f_n$ over $\boldsymbol{C}$. ($\boldsymbol{C}$ means the field of complex numbers.) We know that $0 \leqq \lambda \leqq n-1$. When $\lambda>0$, we say that the system f is degenerate.

In this paper, we discuss some relations between the number "λ" and exceptional linear combinations in X.

For $F \in X$, we set

$$\delta(F)=1-\limsup_{r\to\infty} \frac{N(r, 0, F)}{T(r, f)} ,$$

$$\delta_m(F)=1-\limsup_{r\to\infty} \frac{N_m(r, 0, F)}{T(r, f)} \qquad (m \geqq 1)$$

and $m(F)$=the minimum of multiplicities of all zeros of F ($m(F)=\infty$ when $F(z) \neq 0$), where

$$N_m(r, 0, F)=\sum_{|z_k|<r} \min(m_k, m) \log^+ \frac{r}{|z_k|} + \min(m_0, m) \log r ,$$

$\{z_k\}=\{z \neq 0 ; F(z)=0\}$ and m_k $(\geqq 1)$ is the multiplicity of zero of F at z_k $(k=1, 2, \cdots)$ and m_0 $(\geqq 0)$ is that of F at the origin.

Cartan ([1]) proved

THEOREM A. *If* $\lambda=0$, *then*

1) $\sum_{F \in X} \delta_n(F) \leqq n+1$,

2) *for any* $n+2$ *combinations* $\{F_i\}_{i=0}^{n+1}$ *in* X,