J. Math. Soc. Japan
Vol. 29, No. 1, 1977

On an integer associated with an algebraic group

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(Received Feb. 16, 1976)

§ 1. Introduction.

Let G be a connected (real or complex) Lie group with Lie algebra $\mathfrak{g}$. In general, the exponential map $\exp:\mathfrak{g}\rightarrow G$ is not onto. But recently Markus in [3] and Lai in [2] pointed out that for some algebraic Lie groups G we can associate a natural number q such that for any g in G, the q-th power g^{q} of g lies in $\exp\mathfrak{g}$. In this note, we shall consider an algebraic group theoretic version of these results.

Throughout the paper, k will denote an algebraically closed field (of characteristic 0 or prime). By an algebraic group, we shall mean a linear algebraic group, i. e. a (Zariski) closed subgroup of $GL(m, k)$. The purpose of this note is to prove the following theorem.

THEOREM. *For a given algebraic group G over an algebraically closed field k, we can associate a natural number q such that for any g in G there exists a connected abelian subgroup of G containing g^{q}.*

As a general reference we will presume that the reader is familiar with Borel [1]. The author is pleased to acknowledge his gratitude to F. Grosshans for valuable suggestions and discussions during the preparation of the present paper.

§ 2. char $k=p>0$.

Let G be an algebraic group in $GL(m, k)$. Let g be in G, and let $g=xy=yx$, where x is semisimple and y unipotent, be the Jordan decomposition of g. Let r be the smallest natural number with $p^{r}\geqq m$. We set $q=p^{r}$, and we have $y^{q}=1$, see p. 142 in Borel [1], and so $g^{q}=x^{q}$. Since g^{q} is semisimple, it is contained in some maximal torus, which is connected and abelian.

§ 3. char $k=0$.

Let G_{0} denote the connected component of G containing the identity 1. Then G_{0} is of finite index, say i, in G, and for every $g\in G$ the i-th power g^{i} of g is contained in G_{0}. Hence it suffices to consider connected groups G.